THE MODULI SPACE OF K3 SURFACES

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1. Basic theory of K3 surfaces

We follow [Deb20, §2].

Definition 1.1. A *K3 surface* is a compact surface S such that:

- $\Omega_S^2 \simeq \mathcal{O}_S$; and
- $H^1(S, \mathcal{O}_S) = 0$.

We can readily compute the Euler characteristic of $\mathcal{O}_S$ using Serre duality:

$$\chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - 0 + h^0(S, \mathcal{O}_S) = 2.$$

Example 1.2. The surface $x^4 + y^4 + z^4 + u^4 = 0$ in $\mathbf{P}^3$ is a K3 surface.

Let S be a K3 surface. The exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_S \xrightarrow{\exp} \mathcal{O}_S^\times \to 1$$

gives the long exact sequence in cohomology

$$\begin{aligned} 0 \to & H^0(S, \mathbb{Z}) = \mathbb{Z} \to H^0(S, \mathcal{O}_S) = \mathbb{C} \to H^0(S, \mathcal{O}_S^\times) = \mathbb{C}^\times \\ \to & H^1(S, \mathbb{Z}) \to H^1(S, \mathcal{O}_S) = 0 \to H^1(S, \mathcal{O}_S^\times) = \operatorname{Pic}(S) \\ \to & H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S) \to \cdots . \end{aligned} \tag{1.3}$$

This implies $H^1(S, \mathbb{Z}) = 0$.

Lemma 1.4. *The Picard group of a K3 surface is torsion-free.*

Proof. Suppose $\mathcal{M} \in \operatorname{Pic}(S)$ is torsion. Then by Riemann-Roch,

$$\chi(S, \mathcal{M}) = \chi(S, \mathcal{O}_S) + \frac{1}{2}\mathcal{M}.(\mathcal{M} - K) = \chi(S, \mathcal{O}_S) = 2.$$

By Serre duality $\chi(S, \mathcal{M}) = h^0(S, \mathcal{M}) + h^0(S, \mathcal{M}^{-1}) - h^1(S, \mathcal{M})$, so either $\mathcal{M}$ or $\mathcal{M}^{-1}$ has global sections. But if $\mathcal{M}$ has a global section s, then s is also a global section of $\mathcal{M}^{\otimes m} \simeq \mathcal{O}_S$ for some $m > 0$, so it is nowhere vanishing. Thus $\mathcal{M}$ is also trivial. The same argument works for when $\mathcal{M}^{-1}$ has a global section. □

Thus, by (1.3), the group $H^2(S, \mathbb{Z})$ is torsion-free. By Poincaré duality, so is $H_2(S, \mathbb{Z})$. Now by the universal coefficient theorem $H_1(S, \mathbb{Z})$ is also torsion-free, hence zero (in fact, $\pi_1(S) = 0$). By Poincaré duality $H^3(S, \mathbb{Z}) = 0$, so the entire singular (co)homology of S is torsion-free.

The topological Euler characteristic of S is $c_2(S) = b_2(S)+2$. By Noether's formula $12\chi(S, \mathcal{O}_S) = c_1^2(S)+c_2(S)$ where $c_1^2(S) := (K, K) = 0$, we conclude $c_2(S) = 24$ and $b_2(S) = 22$. Thus $H^2(S, \mathbb{Z})$ is a free abelian group of rank 22. The intersection form on it is even unimodular, and by Hirzebruch's formula it has signature

$$\frac{1}{3}(c_1^2(S) - c_2(S)) = -16.$$

In fact, this implies

$$H^2(S, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

where:

- U is the *hyperbolic plane*, i.e., $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with pairing $(e_i, e_j) = \delta_{ij}$.
- $E_8(-1)$ is $(\Lambda, -q)$ where (Λ, q) is the E_8-lattice.

1.1. **Properties of line bundles.** Let $\mathcal{L}$ be an ample line bundle on S. Then by Kodaira vanishing $H^1(S, \mathcal{L}) = 0$, and $H^0(S, \mathcal{L}^{-1}) = H^2(S, \mathcal{L}) = 0$, hence the Riemann-Roch theorem gives

$$h^0(S, \mathcal{L}) = \chi(S, \mathcal{L}) = \frac{1}{2}\mathcal{L}^2 + 2.$$

We have the following theorem, which allows us to determine when an ample line bundle is globally generated:

Theorem 1.5. *Let $\mathcal{L}$ be an ample line bundle on a K3 surface S. Then $\mathcal{L}$ is generated by global sections if and only if there are no divisors D on S such that $\mathcal{L}.D = 1$ and $D^2 = 0$.*

When $\mathcal{L}$ is globally generated, it defines a morphism

$$\varphi_{\mathcal{L}} \colon S \to \mathbf{P}^g,$$

where $g := \frac{1}{2}\mathcal{L}^2 + 1$. By Bertini's theorem general elements $C \in |\mathcal{L}|$ are smooth irreducible curves of genus g, and the restriction of $\varphi_{\mathcal{L}}$ to C is the canonical map $C \to \mathbf{P}^{g-1}$.

One can show:

Theorem 1.6. *Let $\mathcal{L}$ be an ample line bundle on a K3 surface. The line bundle $\mathcal{L}^2$ is generated by global sections, and $\mathcal{L}^k$ is very ample for all $k \geq 3$ (i.e., $\varphi_{\mathcal{L}}$ is an embedding).*

Now, we make the following fundamental definition:

Definition 1.7. A *polarization* on a K3 surface S is an ample class $\mathcal{L}$ in $\mathrm{Pic}(S)$ which is not divisible, i.e., there does not exist another line bundle $\mathcal{M}$ on S and an integer $m > 1$ such that $\mathcal{L} \simeq \mathcal{M}^{\otimes m}$. The quantity $g := \frac{1}{2}\mathcal{L}^2 + 1$ is the *genus* of a polarized K3 surface $(S, \mathcal{L})$.

Remark 1.8. For an abelian variety A realized as $\mathbb{C}^g/\Lambda$, a *polarization* is an anti-symmetric form $\omega \colon \Lambda \times \Lambda \to \mathbb{Z}$ satisfying certain properties, which corresponds to the Chern class of an ample line bundle as an element of $H^2(A, \mathbb{Z}) = \wedge^2 \hom(\Lambda, \mathbb{Z})$.

We will be interested in looking at the moduli space of polarized K3 surfaces with fixed genus.

2. The moduli space of polarized K3 surfaces

Fix a genus g, and let $(S, \mathcal{L})$ be a polarized K3 surface of genus g. Since $\mathcal{L}^3$ is ample, S embeds into $\mathbf{P}^{9(g-1)+1}$, with fixed Hilbert polynomial $9(g-1)T^2 + 2$. Thus $(S, \mathcal{L})$ defines a point in the Hilbert scheme of close subschemes of $\mathbf{P}^{9(g-1)+1}$ with Hilbert polynomial $9(g-1)T^2 + 2$. The subscheme parametrizing K3 surfaces is open and smooth. The problem now is to construct the quotient by $\mathrm{PGL}(9(g-1)+2)$. It was prove in [Vie90] that:

Theorem 2.1. *Let $g > 1$. Then there exists an irreducible 19-dimensional quasi-projective coarse moduli space $\mathcal{K}_g$ for polarized complex K3 surfaces of genus g.*

We now relate the moduli space $\mathcal{K}_g$ to orthogonal Shimura varieties.

Theorem 2.2. *Let $(S, \mathcal{L})$ and $(S', \mathcal{L}')$ be polarized complex K3 surfaces. If there exists an isometry of lattices*

$$\varphi \colon H^2(S', \mathbb{Z}) \xrightarrow{\sim} H^2(S, \mathbb{Z})$$

such that $\varphi(\mathcal{L}') = \mathcal{L}$ and $\varphi_{\mathbb{C}}(H^{2,0}(S')) = H^{2,0}(S)$, there exists an isomorphism $\sigma \colon S \xrightarrow{\sim} S'$ such that $\varphi = \sigma^*$.

Recall that for a K3 surface S, the singular cohomology $H^2(S,\mathbb{Z})$ is isomorphic to $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. A polarization gives a primitive vector $h_g \in \Lambda_{K3}$ such that $h_g^2 = 2(g-1)$ (in fact, there is a unique $O(\Lambda_{K3})$-orbit of such elements). For example, we can take $h_g = e_1 + (g-1)e_2$. Then

$$\Lambda_{K3,g} := h_g^\perp = U^{\oplus 2} \oplus E_8(-1) \oplus I_1(2-2g),$$

where $I_1(2-2g) = \mathbb{Z}x$ has $x^2 = 2-2g$. Now the *period* is $p(S,\mathcal{L}) := \varphi_{\mathbb{C}}(H^{2,0}(S)) \in \Lambda_{K3} \otimes \mathbb{C}$, which is in $h_g^\perp$, and it satisfies the Hodge-Riemann bilinear relations

$$p(S,\mathcal{L}) \cdot p(S,\mathcal{L}) = 0, \qquad p(S,\mathcal{L}) \cdot \overline{p(S,\mathcal{L})} > 0.$$

Define the 19-dimensional complex manifold

$$\Omega_g := \{[x] \in \mathbf{P}(\Lambda_{K3,g} \otimes \mathbb{C}) : x \cdot x = 0, x \cdot \overline{x} > 0\},$$

so $p(S,\mathcal{L}) \in \Omega_g$. Now, we obtain the *period map*

$$\begin{aligned} \wp_g \colon \mathcal{K}_g &\to \mathcal{P}_g := \mathrm{SO}(\Lambda_{K3,g}) \backslash \Omega_g \\ [(S,\mathcal{L})] &\mapsto [p(S,\mathcal{L})]. \end{aligned}$$

Now, Torelli's theorem can be re-stated as:

Theorem 2.3. *Let $g > 1$. The period map $\wp_g \colon \mathcal{K}_g \to \mathcal{P}_g$ is an open immersion.*

One can even explicitly describe the image:

Proposition 2.4. *Let $g > 1$. The image of $\wp_g$ is the complement of one irreducible hypersurface if $g \not\equiv 2 \pmod 4$ and of two irreducible hypersurfaces if $g \equiv 2 \pmod 4$.*

We see that the quotient $\mathcal{P}_g$ already looks like a Shimura variety! Indeed, Ω_g is the $\mathrm{SO}(2,19)(\mathbb{R})$-conjugacy classes of Hodge structures $h \colon \mathbb{S} \to \mathrm{SO}(2,19)_{\mathbb{R}}$ for which $\pm\psi$ is a polarization and the Hodge numbers are $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} = 19$.

3. Arithmetic aspects

The period map can be enhanced to the following setting: for certain compact open subgroups $\mathbb{K} \subset \mathrm{SO}(2,19)(\mathbf{A}_f)$, define the notion of a level K-structure on K3 surfaces, and let $\mathcal{K}_{g,\mathbb{K}}$ be the moduli space of K3 surfaces of genus g and with level $\mathbb{K}$-structure. The moduli space is a finite étale cover of $\mathcal{K}_g$. Now, for such a compact open subgroup, the period map is a morphism

$$\wp_{g,\mathbb{K}} \colon \mathcal{K}_{g,\mathbb{K}} \otimes \mathbb{C} \to Sh_{\mathbb{K}}(\mathrm{SO}(2,19),\Omega)_{\mathbb{C}}$$

where $Sh_{\mathbb{K}}(\mathrm{SO}(2,19),\Omega)_{\mathbb{C}}$ is the Shimura variety associated to $\mathrm{SO}(2,19)$. Both sides of the morphisms are defined over $\mathbb{Q}$, and the main theorem of [Riz05] states:

Theorem 3.1. *The period morphism $\wp_{g,\mathbb{K}}$ descends to a morphism*

$$\mathcal{K}_{g,\mathbb{K}} \otimes \mathbb{Q} \to Sh_{\mathbb{K}}(\mathrm{SO}(2,19),\Omega).$$

We outline the construction of the period map in the simplest case where $\mathbb{K}_n = \{g \in \mathrm{SO}(2,19)(\widehat{\mathbb{Z}}) : g \equiv 1 \pmod n\}$.

Definition 3.2. Let $(\pi \colon X \to S, \mathcal{L})$ be a polarized K3 space of genus g. Then a level $\mathbb{K}_n$-structure is an isomorphism α_n between the orthogonal complement of $c_1(\mathcal{L})$ in $R^2_{et}\pi_*(\mathbb{Z}/n\mathbb{Z})(1)$ with $\Lambda_{K3,g}$. The moduli space of polarized K3 surfaces of genus g with a level $\mathbb{K}_n$-structure is denoted $\overline{\mathcal{K}_{g,\mathbb{K}_n}}$.

On the other hand, the Shimura variety $Sh_{\mathbb{K}_n}(\mathrm{SO}(2,19),\Omega_g) = G(\mathbb{Q})\backslash\Omega_g \times G(\mathbf{A}_f)/\mathbb{K}_n$ admits an interpretation as the moduli space of 4-tuples $((W,h), s, \alpha\mathbb{K}_n)$ where:

1. (W,h) is a orthogonal space over $\mathbb{Q}$ isomorphic to the quadratic space with quadratic form $-x_1^2 - x_2^2 + x_3^2 + \cdots + x_{20}^2 + (g-1)x_{21}^2$

(2) $s\colon \mathbb{S} \to \mathrm{SO}(W,h)$ is a Hodge structure with $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} = 19$
(3) Orthogonal isomorphisms $\alpha\colon V_{2d} \otimes \mathbb{A}_f \to W \otimes \mathbb{A}_f$, modulo $\mathbb{K}_n$.

Now, to define $\mathcal{K}_{g,\mathbb{K}_n} \to Sh_{\mathbb{K}_n}(\mathrm{SO}(2,19), \Omega_g^{\pm})$, we take a polarized K3 surface $(X, \mathcal{L})$ and a level structure $\alpha\colon H^2(S, \mathbb{Z}/n)(1) \simeq \Lambda_{K3,g} \otimes \mathbb{Z}/n$. Then we may choose a lift of α to an isomorphism $\widetilde{\alpha}\colon H^2(S, \widehat{\mathbb{Z}})(1) \simeq \Lambda_{K3,g} \otimes \widehat{\mathbb{Z}}$, which is now well-defined up to $\mathbb{K}_n$.

References

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