PERIODS DETECTING EISENSTEIN SERIES AND SUMS OF L-VALUES I

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Abstract. We study the automorphic period associated to a G-Hamiltonian variety M whose dual is $\check{M} = T^*(\check{G}/\check{L})$, where $\check{G}$ is a general linear group and $\check{L}$ is a Levi subgroup. For certain cuspidal Eisenstein series, we prove that their period is equal to a finite sum of special values of L-functions. This sum is indexed by the fixed points of the associated extended L-parameter on $\check{M}$, confirming a conjecture by Ben-Zvi-Sakellaridis-Venkatesh in this case.

Contents

1. Introduction

1.1. **Relative Langlands conjectures of BZSV.** The relative Langlands program investigates the relation between periods of automorphic forms and special values of L-functions. In their seminal paper [BSV24], Ben–Zvi, Sakellaridis and Venkatesh proposed a general framework for this relationship. Their central idea is that periods are associated with Hamiltonian G-varieties, and each Hamiltonian variety should admit a "dual" $\check{M}$, which is a $\check{G}$-Hamiltonian variety. With this framework, the period $\mathcal{P}_M$ associated to M is conjectured to an L-value attached to $\check{M}$.

To formulate a precise conjecture, the authors of [BSV24] work in the context of function fields with everywhere unramified data. A part of their conjecture [BSV24, Conjecture 14.2.1] can be summarized as follows:

Conjecture 1.1.1 (Ben-Zvi-Sakellaridis-Venkatesh)**.** *Assume that $\check{M} = T^*\check{X}$, where $\check{X}$ is a $\check{G}$-spherical variety. Let π be a tempered, everywhere unramified automorphic representation of $G(\mathbb{A})$ with L-parameter ϕ. If ϕ has only finitely many fixed points $\{x_1, \cdots, x_r\}$ on $\check{X}$. Then for a suitably normalized spherical vector $f \in \pi$, we have*

$$\mathcal{P}_M(f) \sim \sum_i L(0, (T_{x_i}\check{X})^{\not{\partial}})$$

In the number field setting, Mao, Wan and Zhang [MWZ24] formulated an analog of the Conjecture 1.1.1, under the assumption that the hypothetical extended L-parameter of π only has at most one fixed point on $\check{M}$. The goal of this paper is to prove specific cases of Conjecture 1.1.1 in the style of [MWZ24] for number fields. We focus on the case where $\check{X} = \check{G}/\check{L}$, with $\check{G}$ a general linear group and $\check{L}$ a Levi subgroup. A key feature of this case is that the set of fixed points is not necessarily a singleton, leading to an equality of the form

$$\text{Automorphic period} \quad \text{“} = \text{”} \quad \text{Sum of } L\text{-values} \tag{1.1.1}$$

We note that the related work [Wan24] of Wan, who gavemain another example of (1.1.1), that the period associated to $\mathrm{U}(2)\backslash \mathrm{SO}(5)$ equals to a sum of two L-values.

1.2. **The main result.**

1.2.1. *The period.* Throughout the article, we fix a number field F. Let $\mathbb{A} := \mathbb{A}_F$ and fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$. We denote the general linear group GL_k over F by G_k.

For the introduction, we fix integers $n > 0$ and $m > 0$. Let $G = G_{2n+m}$. Let N be the upper triangular unipotent subgroup of G. Let Q be the standard parabolic subgroup of G with Levi component $G_{2n+1} \times G_1^{m-1}$. Denote the unipotent radical of Q by U. We define a character ψ_U of $U(\mathbb{A})$ by

$$\psi_U(u_{ij}) = \psi(u_{2n+1,2n+2} + \cdots + u_{2n+m-1,2n+m}).$$

Let H denote the symplectic group Sp_{2n} preserving the symplectic form $\begin{pmatrix} & J \\ -J & \end{pmatrix}$ with $J = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$. We embed H into G as a subgroup in the upper-left corner. Note that H normalizes U, and the character ψ_U is invariant under the conjugation action of $H(\mathbb{A})$.

For an automorphic form f on $G(\mathbb{A})$, we define its Fourier coefficient along U by

$$f_{U,\psi}(g) := \int_{[U]} f(ug)\psi_U^{-1}(u)\mathrm{d}u$$

We then define the period integral

$$\mathcal{P}(f) := \int_{[H]} f_{U,\psi}(h)\mathrm{d}h. \tag{1.2.1}$$

We remark that the integral defining $\mathcal{P}$ is not necessarily absolutely convergent. Thus, to define this period for a broad class of automorphic forms, a regularization of the integral (1.2.1) is required.

The period $\mathcal{P}$ is the period associated to the G-hyperspherical Hamiltonian variety $M = T^*(G/HU, \psi_U)$, whose conjectural dual variety is $\check{M} = T^*\check{X}$, where $\check{X} = G_{2n+m}/G_n \times G_{n+m}$ (see [BSV24, §4], [Sak13, Appendix A]).

1.2.2. *The main result.* We will study the period $\mathcal{P}(f)$ when f is a cuspidal Eisenstein series. Let P be a standard parabolic subgroup of G, let π be a unitary cuspidal automorphic representation of M_P. For concreteness, we assume $M_P = G_{n_1} \times \cdots \times G_{n_k}$ and $\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$, where each π_i is a unitary cuspidal automorphic representation of G_{n_i}.

Let $A_{M_P}^\infty \cong \mathbb{R}_{>0}^k$ denote the central subgroup of $M_P(\mathbb{A})$, consisting of elements of the form $\begin{pmatrix} t_1 I_{n_1} & & \\ & \cdots & \\ & & t_k I_{n_k} \end{pmatrix}$ with $t_i \in \mathbb{R}_{>0}$.

We write π_0 for the unique unramified twist of π such that the central character of π_0 is trivial on $A_{M_P}^\infty$. By an unramified twist, we mean π_0 is of the form $\pi_1|\cdot|^{\lambda_1} \boxtimes \cdots \boxtimes \pi_k|\cdot|^{\lambda_k}$ with $\lambda_j \in i\mathbb{R}$ for $1 \le j \le k$.

Let $\varphi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$, which we regard as a function on $N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A})$. Let $E(\varphi) := E(\cdot, \varphi, 0)$ be the associated Eisenstein series. We denote by $\mathrm{Fix}(\pi)$ the fixed point set of the (hypothetical) L-parameter of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$ acting on $\check{X}$.

Theorem 1.2.1 (Theorem 6.4.2, rough form)**.** *Assume that* $\mathrm{Fix}(\pi_0)$ *is finite, then we have* $\mathrm{Fix}(\pi_0) = \mathrm{Fix}(\pi)$ *and*

(1) The period $\mathcal{P}(E(\varphi))$ *can be defined canonically.*
(2) Let $\mathtt{S}$ *be a sufficiently large set of places of* F*, then we have*

$$\begin{aligned}\mathcal{P}(E(\varphi)) =& (\Delta_H^{\mathtt{S},*})^{-1} \prod_{1 \le i < j \le k} L(1, \pi_i \times \pi_j^\vee)^{-1} \times \\ & \sum_{\sigma \in \mathrm{Fix}(\pi)} L^{\mathtt{S}}(1, T_\sigma \check{X}) \cdot (\textit{local zeta integral of } W_{\varphi,\mathtt{S}}^{M_P} \textit{ at } \mathtt{S}),\end{aligned} \tag{1.2.2}$$

where
- $\Delta_H^{\mathtt{S},*}$ *is a constant related to the Tamagawa measure (see §2.2.1)*
- *We define*

$$W_\varphi^{M_P}(g) := \int_{[M_P^N]} \varphi(ug)\psi^{-1}(u)\mathrm{d}u$$

to be the Whittaker coefficient of the section φ, where $M_P^N = M_P \cap N$. And we decompose $W_\varphi^{M_P}$ as $W_\varphi^{M_P} = W_\varphi^{M_P,\mathtt{S}} W_{\varphi,\mathtt{S}}^{M_P}$ with $W_\varphi^{M_P,\mathtt{S}}$ is spherical and equals 1 at $g = 1$.

We will address the definition of $\mathcal{P}(E(\varphi))$ in §1.3. For now, let us focus on the identity (1.2.2).

Remark 1.2.2.
- If $\pi_i = \pi_j$ for some $i \neq j$, the L-function $L(s, \pi_i \times \pi_j^\vee)$ has a pole at $s = 1$. In this case, the right-hand side of (1.2.2) is interpreted as 0, so $\mathcal{P}(E(\varphi)) = 0$.
- Consider the function field analogue of Theorem 1.2.1 under the assumption that all data are unramified. We may take $\mathtt{S} = \varnothing$. Then if we normalize so that $W_\varphi^{M_P}(1) = 1$, equation (1.2.2) becomes
$$\mathcal{P}(E(\varphi)) = \Delta_H^{*,-1} \prod_{1 \le i < j \le k} L(1, \pi_i \times \pi_j^\vee) \sum_{\sigma \in \mathrm{Fix}(\pi)} L(1, T_\sigma \check{X}).$$
By a result of Shahidi [Sha81, §4], the Whittaker coefficient $W_{E(\varphi)}(g) := \int_{[N]} E(\varphi)(u) \psi^{-1}(u) \mathrm{d}u$ of $E(\varphi)$ satisfies $W_{E(\varphi)}(1) = \prod_{1 \le i < j \le k} L(1, \pi_i \times \pi_j^\vee)^{-1}$. Therefore, if we normalize so that $W_{E(\varphi)}(1) = 1$, then
$$\mathcal{P}(E(\varphi)) = \Delta_H^{*,-1} \sum_{\sigma \in \mathrm{Fix}(\pi)} L(1, T_\sigma \check{X}).$$
This is the form which exactly looks like Conjecture 1.1.1.
- The L function $L(1, T_\sigma \check{X})$ is an example of *non-linear* L-function; see [BSV24, Remark 14.2.4], [CV24].
- A version of Theorem 1.2.1 also holds for $\check{X} = \mathrm{GL}_{2n} / \mathrm{GL}_n \times \mathrm{GL}_n$ (see §5). In this case, the period associated to the dual variety $\check{M}$ (rather than M) is the *Friedberg-Jacquet period* (or *linear period*) studied in [FJ93].
- In our earlier work [LWX25], we proved a special case of Theorem 1.2.1 when $m = 1$ and P is a maximal parabolic. We note that the method in the present article differs from the that of *loc .cit.*. We also note that the period associated to $\check{M}$ is also studied in [FJ93].

1.2.3. *A more precise formulation.* The statement of Theorem 1.2.1 involves the hypothetical global Langlands correspondence. To avoid this, we now describe the fixed point set $\mathrm{Fix}(\pi)$ and the L-function $L(1, T_\sigma \check{X})$ solely in terms of the representation π. In Appendix §A, we verify that, assuming global Langlands correspondence, this description coincides with the definition given by the L-parameter.

Let P be a standard parabolic and let π be a unitary cuspidal automorphic representation of M_P as above. The condition for $\mathrm{Fix}(\pi)$ to be discrete is equivalent to the following:

(1.2.3) For any subset $I \subset \{1, 2, \cdots, k\}$ such that $\sum_{i \in I} n_i = n$, we have $\pi_i \neq \pi_j$ for any $i \in I$ and $j \in I^c$, where I^c denote the complement of I.

From this description, we see that the condition that $\mathrm{Fix}(\pi_0)$ is discrete is stronger than the condition that $\mathrm{Fix}(\pi)$ is discrete. Therefore, Theorem 1.2.1 is slightly weaker than the expectation from Conjecture 1.1.1.

Henceforth, we assume that (1.2.3) holds for π_0. We define $\mathrm{Fix}(\pi)$ as the set of permutations $\sigma : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$, for which there exists $1 \le t \le k$ with:

(1) $n_{\sigma(1)} + \cdots + n_{\sigma(t)} = n, n_{\sigma(t+1)} + \cdots + n_{\sigma(k)} = n + m$.
(2) $\sigma(1) < \cdots < \sigma(t)$ and $\sigma(t+1) < \cdots < \sigma(k)$.

Note that t is uniquely determined by σ and $\mathrm{Fix}(\pi)$ is in bijection with the set

$$\left\{ I \subset \{1, 2, \cdots, k\} \mid \sum_{i \in I} n_i = n \right\}.$$

In particular $\mathrm{Fix}(\pi)$ is non-empty if and only if, up to permutation, $E(\varphi)$ is an Eisenstein series "passing through" the maximal Levi subgroup $G_n \times G_{n+m}$. In other words, the period $\mathcal{P}$ *detects* $(n, n+m)$*-Eisenstein series.*

For $\sigma \in \mathrm{Fix}(\pi)$ corresponds to the subset I above, we put

$$L(s, T_\sigma \check{X}) = \prod_{(i,j) \in I \times I^c} L(s, \pi_i^\vee \times \pi_j) L(s, \pi_i \times \pi_j^\vee).$$

Note that the condition (1.2.3) ensures that this L-function does not have a pole at $s = 1$. This completes the description of $\mathrm{Fix}(\pi)$ and $L(1, T_\sigma \check{X})$ in (1.2.2).

We now describe the local zeta integral. Let v be a place of F and let $R = M_R N_R$ a standard parabolic subgroup of G. Let Π be an irreducible generic representation of M_R and let $\mathcal{W}(\Pi, \psi_v)$ denote the Whittaker model of Π_R. We define $\mathrm{Ind}_{R(F_v)}^{G(F_v)} \mathcal{W}(\Pi, \psi_v)$ to be the space of functions $W^{M_R} : G(F_v) \to \mathbb{C}$ such that for any $g \in G(F_v)$, the map $m \in M_R(F_v) \mapsto \delta_R^{-1}(m) W^{M_R}(mg)$ belongs to $\mathcal{W}(\Pi, \psi_v)$.

Let Q_n denote the standard parabolic subgroup of G with Levi component $G_n \times G_{n+m}$. Let $\Pi_M = \Pi_n \boxtimes \Pi_{n+m}$ be an irreducible generic representation of $M_{Q_n}(F_v)$. For $W^M \in \mathrm{Ind}_{Q_n(F_v)}^{G(F_v)} \mathcal{W}(\Pi_M, \psi_v)$ and $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, we define

$$Z_v(\lambda, W^M) = \int_{N_H(F_v) \backslash H(F_v)} W^M(h) e^{\langle \lambda, H_{Q_n}(h) \rangle} \mathrm{d}h,$$

where $N_H := N \cap H$. The integral is convergent for $\mathrm{Re}(\lambda)$ lies in a suitable cone and has meromorphic continuation to $\mathfrak{a}_{Q_n,\mathbb{C}}^*$.

Note that $\mathrm{Fix}(\pi)$ can be identified with a subset of the Weyl group W^G of G. Specifically, we identify an element $\sigma \in \mathrm{Fix}(\pi)$ with the permutation that preserves the internal order of each block of M_P. We write P_σ for the standard parabolic subgroup of G_{2n+m} with $M_{P_\sigma} = G_{n_{\sigma(1)}} \times \cdots \times G_{n_{\sigma(k)}}$. Let $\mathtt{S}$ be a finite set of places of F. Then σ induces a *normalized intertwining operator* (see §2.4.3) $N_{\pi,\mathtt{S}} : \mathrm{Ind}_{P(F_\mathtt{S})}^{G(F_\mathtt{S})} \mathcal{W}(\pi, \psi_\mathtt{S}) \to \mathrm{Ind}_{P_\sigma(F_\mathtt{S})}^{G(F_\mathtt{S})} \mathcal{W}(\sigma\pi, \psi_\mathtt{S})$.

Let $\Pi_{\sigma,n} = \pi_{\sigma(1)} \boxplus \cdots \boxplus \pi_{\sigma(t)}$ (parabolic induction) and $\Pi_{\sigma,n+m} = \pi_{\sigma(t+1)} \boxplus \cdots \boxplus \pi_{\sigma(k)}$. Finally, let $\Omega_{\mathtt{S}}^{Q_n}$ denote the *Jacquet integral* (see §2.6.2), it is a map from $\mathrm{Ind}_{P_\sigma(F_{\mathtt{S}})}^{G(F_{\mathtt{S}})} \mathcal{W}(\sigma\pi, \psi_{\mathtt{S}})$ to $\mathrm{Ind}_{Q_n(F_{\mathtt{S}})}^{G(F_{\mathtt{S}})} \mathcal{W}(\Pi_{\sigma,n} \boxtimes \Pi_{\sigma,n+m}, \psi_{\mathtt{S}})$.

With these notations, the precise form of the identity (1.2.2) is given by

$$\begin{aligned}\mathcal{P}(E(\varphi)) =& (\Delta_H^{\mathtt{S},*})^{-1} \prod_{1 \le i < j \le k} L(1, \pi_i \times \pi_j^\vee)^{-1} \times \\ & \sum_{\sigma \in \mathrm{Fix}(\pi)} L^{\mathtt{S}}(1, T_\sigma \check{X}) \left(\prod_{1 \le i < j \le k} L_{\mathtt{S}}(1, \pi_{\sigma(i)} \times \pi_{\sigma(j)}^\vee) \right) Z_{\mathtt{S}}(0, \Omega_{\mathtt{S}}^{Q_n}(N_{\pi,\mathtt{S}}(\sigma) W_{\varphi,\mathtt{S}}^{M_P})).\end{aligned} \tag{1.2.4}$$

1.3. **Definition of the period.** We now discuss the definition of the period integral.

1.3.1. *Definition via continuous extension.* Let $\mathcal{S}([G])$ denote the space of Schwartz functions on $[G]$ and let $\mathcal{T}([G])$ denote the space of smooth functions of uniform moderate growth on $[G]$ (see 2.3.1). Both of them are vector spaces over $\mathbb{C}$ carrying a natural topology. When $f \in \mathcal{S}([G])$, the integral defining $\mathcal{P}(f)$ is absolutely convergent.

Let $\mathfrak{X}(G)$ denote the set of cuspidal datum of G. (see §2.3.1) We have the following coarse Langlands spectral decomposition according to cuspidal support:

$$L^2([G]) = \widehat{\bigoplus}_{\chi \in \mathfrak{X}(G)} L^2_\chi([G]).$$

For a subset $\mathfrak{X} \subset \mathfrak{X}(G)$, we put $L^2_{\mathfrak{X}}([G]) = \widehat{\bigoplus}_{\chi \in \mathfrak{X}} L^2_\chi([G])$, and let $\mathcal{S}_{\mathfrak{X}}([G]) = \mathcal{S}([G]) \cap L^2_{\mathfrak{X}}([G])$. These are Schwartz functions with cuspidal support in $\mathfrak{X}$. Let $\mathcal{T}_{\mathfrak{X}}([G])$ denote the orthogonal complement of $\mathcal{S}_{\mathfrak{X}^c}([G])$ in $\mathcal{T}([G])$. When $\mathcal{T}_{\mathfrak{X}}([G])$ carries the subspace topology inherited from $\mathcal{T}([G])$, $\mathcal{S}_{\mathfrak{X}}([G])$ is a dense subspace of $\mathcal{T}_{\mathfrak{X}}([G])$.

Let $\mathfrak{X}_\Delta$ denote the cuspidal datum represented by (M_P, π) such that π satisfies (1.2.3). We write $\mathcal{S}_\Delta([G])$ (resp. $\mathcal{T}_\Delta([G])$) for $\mathcal{S}_{\mathfrak{X}_\Delta}([G])$ (resp. $\mathcal{T}_{\mathfrak{X}_\Delta}([G])$). Then we have the following theorem:

Theorem 1.3.1. *The period $\mathcal{P}$, defined on $\mathcal{S}_\Delta([G])$, admits a (necessarily unique) continuous extension to $\mathcal{T}_\Delta([G])$.*

Let P be a standard parabolic subgroup of G and let π be a unitary cuspidal automorphic representation of M_P such that π_0 satisfies (1.2.3). Then the Eisenstein series $E(\varphi)$ lies in $\mathcal{T}_\Delta([G])$. This explains the meaning of (1) in Theorem 1.2.1.

1.3.2. *Definition via truncation operator.* When $m = 1$, there is an alternative definition of the period with potential applications, for example, in relative trace formulas. The period $\mathcal{P}$ is taking a Sp_{2n} period of an automorphic form on GL_{2n+1}. The work of Zydor [Zyd19] suggests a regularization of the period $\mathcal{P}$ via truncation. Let $f \in \mathcal{T}([G])$ and let T be a truncation parameter, in *loc. cit.*, Zydor defines a truncated function $\Lambda^T f$ on $[H]$ which is rapidly decreasing. In §7, we prove the following result:

Proposition 1.3.2. *For $f \in \mathcal{T}_\Delta([G])$, the integral*

$$\int_{[H]} \Lambda^T f(h) \mathrm{d}h$$

is independent of T. Moreover, this constant coincides with $\mathcal{P}(f)$ as defined in Theorem 1.3.1.

1.4. **The strategy of the proof.** The proof of Theorem 1.2.1 and Theorem 1.3.1 proceed via an unfolding argument, analogous to the standard unfolding of period integrals into integrals of Whittaker functions.

Let ψ_n be the degenerate character on $N(\mathbb{A})$ defined by

$$\psi_n(u) = \psi(u_{1,2} + \cdots + u_{n-1,n} + u_{n+1,n+2} + \cdots + u_{2n+m-1,2n+m}).$$

For $f \in \mathcal{T}([G])$, we define the associated degenerate Whittaker coefficient by:

$$V_f(g) = \int_{[N]} f(ug)\psi_n^{-1}(u)\mathrm{d}u.$$

The key step is the following proposition:

Proposition 1.4.1 (Proposition 6.2.1)**.** *For $f \in \mathcal{S}_\Delta([G])$, then we have*

$$\mathcal{P}(f) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} V_f(h)\mathrm{d}h.$$

The proof of this proposition involves performing a Fourier expansion along certain abelian unipotent subgroups, similar to the Fourier expansion of a cusp form. However, since f is not necessarily cuspidal, extra terms appear in the unfolding process. Our assumption on the cuspidal support of f ensures that these extra terms do not contribute to the period.

For $f \in \mathcal{T}([G])$ and $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$, we define a global zeta integral by

$$Z(\lambda, f) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} V_f(h) e^{\langle \lambda, H_{Q_n}(h)\rangle}\mathrm{d}h.$$

The global zeta integral $Z(\lambda, f)$ is absolutely convergent when $\mathrm{Re}(\lambda)$ lies in suitable half-plane. We then show that for $f \in \mathcal{T}_\Delta([G])$, the zeta integral $Z(\lambda, f)$ is holomorphic at $\lambda = 0$, and the map $f \mapsto Z(0, f)$ provides the continuous extension of $\mathcal{P}$ to $\mathcal{T}_\Delta([G])$.

Let $Q_n^H := Q_n \cap H$ be the Siegel parabolic of H. Let K_H denote a maximal compact of $H(\mathbb{A})$ which is in good position relative to the upper triangular Borel at the non-Archimedean places. Using the Iwasawa decomposition $H(\mathbb{A}) = Q_n^H(\mathbb{A})K_H$, the zeta integral $Z(\lambda, f)$ can be expressed as

$$Z(\lambda, f) = \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + 1, R(k) f_{Q_n})\mathrm{d}k, \tag{1.4.1}$$

where

(1) $\widetilde{Z}^{\mathrm{RS}}$ denotes the (twisted) Rankin-Selberg zeta integral: for $f \in \mathcal{T}([G_n \times G_{n+m}])$, we define

$$\widetilde{Z}^{\mathrm{RS}}(s,f) = \int_{N_n(\mathbb{A})\backslash G_n(\mathbb{A})} W_f(J^t g^{-1} J, \begin{pmatrix} g & \\ & 1_m \end{pmatrix}) |\det g|^s \mathrm{d}g,$$

where N_n denotes the upper triangular unipotent subgroup of G_n and

$$W_f(g) = \int_{[N_n \times N_{n+m}]} f(ug)\psi^{-1}(u)\mathrm{d}u$$

is the Whittaker coefficient of f.

(2) $\alpha \in \Delta_{Q_n}$ is the unique simple root and $\alpha^\vee$ denotes its coroot and $s_\lambda = -\langle \lambda, \alpha^\vee \rangle$.

By (1.4.1), the problem reduces to show that the Rankin-Selberg integral admits a continuous extension to uniform moderate growth functions with specific cuspidal support. When $m = 1$, this is achieved in [BCZ22, §7]. We will show the case when $m > 1$ in §4. The proof involves another unfolding process and an application of the Phragmen-Lindelöf principle.

Finally, when $f = E(\varphi)$ is a cuspidal Eisenstein series, we use the formula (1.2.4) to compute $\mathcal{P}(E(\varphi)) = Z(0,f)$. By combining the constant term formula for Eisenstein series and the Euler decomposition of Rankin-Selberg integral, we will achieve (1.2.2). The summation of L-values appearing in the formula results from the formula for the constant terms of Eisenstein series.

1.5. **The structure of this article.** After the preliminaries in §2, we will review the result of canonical extension of Rankin-Selberg period of corank 1 [BCZ22, §7] and equal rank [BCZ22, §10.3] to functions with certain cuspidal support in §3. And we will do the higher corank case in §4. Then we will study the period detecting (n,n)-Eisenstein series in §5 and detecting $(n, n+m)$-Eisenstein series in §6.

1.6. **Acknowledgement.** We thank Chen Wan for introducing this problem. We thank Raphaël Beuzart-Plessis, Paul Boisseau, Colin Loh, Omer Offen, Dihua Jiang, Zeyu Wang, Hang Xue, Lei Zhang and Wei Zhang for helpful suggestions and discussions.

2. Preliminaries

2.1. **General notations.**

- Throughout this article, unless otherwise specified, we fix a number field F. Let $\mathbb{A} := \mathbb{A}_F$ be the adèle ring of F and let $\mathbb{A}_f$ be the finite adèles. Let v be a place of F, we write F_v for the completion. Let $\mathtt{S}$ be a finite set of places of F, we write $F_{\mathtt{S}} := \prod_{v \in \mathtt{S}} F_v$.
- We write G_n for the general linear group GL_n over F. Let $\mathtt{S}_\infty$ be the set of Archimedean places, we write $F_\infty := F_{\mathtt{S}_\infty}$.

- Let Sp_{2n} be the symplectic group preserving the symplectic form $\begin{pmatrix} & J \\ -J & \end{pmatrix}$ with $J = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$.
- For integer $m > 1$, let 1_m denote the identity matrix of size m.
- Let $\mathcal{H}_{>C} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > C\}$.
- For a ring R, we write R^n the column vector with coefficient in R of size n and we write R_n for the row vector of size n.
- Let f, g be two positive functions on a set X, we write $f \ll g$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ for any $x \in X$.
- For a set X and a subset $A \subset X$, we write A^c the complement of A in X.

2.2. **Groups.** Let G be a connected linear algebraic group over a global field F. Let $[G] := G(F)\backslash G(\mathbb{A})$ the adèlic quotient of G.

2.2.1. *Tamagawa measure.* We fix the Tamagawa measure dg on $G(\mathbb{A})$, and thus on $[G]$ as described in [BCZ22, section 2.3]. We recall the definition here. Fix a non-trivial additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$. We decompose ψ as $\psi = \prod \psi_v$. For each place v of F, ψ_v determines the self-dual measure on F_v. Let ω be an F-rational G-invariant top differential form on G. For each place v, $|\omega|_v$ gives a measure $\mathrm{d}g_v$ on $G(F_v)$. Moreover, according to the results of Gross [Gro97], there exists a global Artin-Tate L-function $L_G(s)$ such that

$$\mathrm{d}g_v(G(\mathcal{O}_v)) = L_{G,v}(0)$$

for almost all places v. We denote by

$$\Delta_{G,v} := L_{G,v}(0)$$

and let Δ_G^* denote the leading coefficient of the Laurent expansion of $L_G(s)$ at $s = 0$. The Tamagawa measure is defined by

$$\mathrm{d}g = (\Delta_G^*)^{-1} \prod_v \mathrm{d}g_v.$$

The measure is independent of the choice of ω. For a finite set $\mathtt{S}$ of places of F, let $\Delta_G^{\mathtt{S},*}$ denote the leading coefficient of the partial L-function $L_G^{\mathtt{S}}(s)$ at $s = 0$.

2.2.2. *Norms and heights.* Let N be a positive integer. For $x \in \mathbb{A}^N$, we define

$$\|x\| = \prod_v \max\{|x_{1,v}|_v, \cdots, |x_{n,v}|_v, 1\},$$

where the product runs over the set of places of F. For a linear algebraic group G, we fix a closed embedding ι of G into an affine space. Then for $g \in G(\mathbb{A})$ we define $\|g\|_{G(\mathbb{A})} = \|\iota(g)\|$. Let $\|\cdot\|'_{G(\mathbb{A})}$ be the norm defined by another embedding ι', then there exists $r > 0$ such that $\|g\|_{G(\mathbb{A})} \ll \|g\|'^{,r}_{G(\mathbb{A})}$. We refer the reader to [Beu21, Appendix A] for more properties of the norm $\|\cdot\|_{G(\mathbb{A})}$.

For the rest of §2.2, we assume that G is a connected reductive group. We fix a maximal split torus A_0 of G and fix a minimal parabolic subgroup P_0 of G containing A_0. A parabolic subgroup of G is called *standard* if it contains P_0 and is called *semi-standard* if it contains A_0.

Let P be a semi-standard parabolic subgroup, we denote by M_P the Levi subgroup of P containing A_0 and denote by N_P the unipotent radical of P. Since the natural map $M_P \times N_P \to P$ is an isomorphism of varieties. We see that $\|mn\|_{P(\mathbb{A})} \sim \|m\|_{M_P(\mathbb{A})}\|n\|_{N_P(\mathbb{A})}$. That is, there exists $c > 1$ such that

$$\|mn\|_{P(\mathbb{A})}^{1/c} \ll \|m\|_{M(\mathbb{A})}\|n\|_{N(\mathbb{A})} \ll \|mn\|_{P(\mathbb{A})}^{c}$$

holds for all $m, n \in M_P(\mathbb{A}) \times N(\mathbb{A})$. As a consequence

(2.2.1) There exists $C > 0$ and $r > 0$ such that for any $g \in G(\mathbb{A})$ and $(m, n, k) \in M_P(\mathbb{A})N_P(\mathbb{A})K$ such that $g = nmk$, we have $\|m\|_{M(\mathbb{A})} \leq C\|g\|_{G(\mathbb{A})}^r$

For a semi-standard parabolic subgroup P of G, we put

$$[G]_P := N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}).$$

We define a norm on $[G]_P$ by

$$\|g\|_P := \inf_{\gamma \in N_P(\mathbb{A})M_P(F)} \|\gamma g\|.$$

2.2.3. *Weyl groups.* Let W be the Weyl group of (G, A_0), that is, the quotient by $M_0(F)$ of the normalizer of A_0 in $G(F)$. For a standard parabolic subgroup P, we write $W^P := W^{M_P}$, and we regarded it as a subgroup of W. For standard parabolic subgroups P, Q, we denote by

$${}_QW_P := \{w \in W \mid M_P \cap w^{-1}P_0w = M_P \cap P_0, \quad M_Q \cap wP_0w^{-1} = M_Q \cap P_0\}.$$

The set ${}_QW_P$ forms a representative of the double coset $W^Q\backslash W/W^P$. For $w \in {}_QW_P$, $M_P \cap w^{-1}M_Qw$ is the Levi factor of the standard parabolic subgroup $P_w = (M_P \cap w^{-1}Qw)N_P$. In the same way, $M_Q \cap wM_Pw^{-1}$ is the Levi factor of the standard parabolic subgroup $Q_w = (L \cap wPw^{-1})N_Q$. We have $P_w \subset P$, $Q_w \subset Q$ We also define

$$W(P; Q) = \{w \in {}_QW_P \mid M_P \subset w^{-1}M_Qw\}.$$

and

$$W(P, Q) = \{w \in {}_QW_P \mid M_P = w^{-1}M_Qw\}.$$

P and Q are called *associate* if $W(P, Q) \neq \varnothing$. For example, for any P, Q and $w \in {}_QW_P$, the parabolics P_w and Q_w are associate.

2.2.4. For a semi-standard parabolic subgroup P of G, define

$$\mathfrak{a}_P^* := X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_P := \mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{R}).$$

We endow $\mathfrak{a}_P$ with the Haar measure such that the lattice $\mathrm{Hom}(X^*(P), \mathbb{Z})$ has covolume 1.

Let $\mathfrak{a}_0 := \mathfrak{a}_{P_0}$ and $\mathfrak{a}_0^* := \mathfrak{a}_{P_0}^*$.

$$\epsilon_P := (-1)^{\dim \mathfrak{a}_P - \dim \mathfrak{a}_G}.$$

Let A_P denote the maximal central split torus of M_P. Then $\mathfrak{a}_P$ can also be identified with $X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$. When $P \subset Q$ are two semi-standard parabolic subgroups, then natural maps $P \hookrightarrow Q$ and $A_Q \hookrightarrow A_P$ induce a projection $\mathfrak{a}_P^* \to \mathfrak{a}_Q^*$ and an injectiion $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$.

Let P_0' be a minimal semi-standard parabolic subgroup, let $\Delta_{P_0'} \subset \mathfrak{a}_{P_0'}$ be the set of simple roots of the A_{P_0} action on $\mathrm{Lie}(N_{P_0'})$. Let P be a semi-standard parabolic subgroup, choose a minimal parabolic subgroup $P_0' \subset P$, then we denote by Δ_P the image of $\Delta_{P_0'}$ under the projection $\mathfrak{a}_{P_0'} \to \mathfrak{a}_P$. Δ_P can also be identified with the set of simple roots of A_P action on $\mathrm{Lie}(N_P)$, in particular, Δ_P is independent of the choice of P_0'.

2.2.5. *Iwasawa decomposition.* Let K be a maximal compact subgroup of $G(\mathbb{A})$ which is in good position with respect to P_0. Then for any semi-standard parabolic subgroup P of G, we have the *Iwasawa decomposition* $G(\mathbb{A}) = P(\mathbb{A})K$.

When $G = G_n$, we denote by K_n the usual maximal compact subgroup of $G_n(\mathbb{A})$. In the main text, we will sometimes use H to denote the symplectic group Sp_{2n}, and K_H will denote the usual maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$ accordingly.

Lemma 2.2.1. *There exists measurable maps $G(\mathbb{A}) \to P(\mathbb{A}) \times K, g \mapsto (p(x), k(x))$ such that for any $g \in G(\mathbb{A})$, we have $g = p(g)k(g)$.*

Proof. Since $P(\mathbb{A}) \times K$ is a Polish space, this follows from Kuratowski and Ryll-Nardzewski measurable selection theorem applied to the natural map $P(\mathbb{A}) \times K \to G(\mathbb{A})$. □

We will sometimes refer to any function $p(g), k(g)$ as in the previous lemma a measurable (family of) Iwasawa decomposition.

For positive integers k, n, denote by $\mathrm{Mat}_{k\times n}(\mathbb{A})$ the set of matrices of size $k \times n$ with coefficients in $\mathbb{A}$. For future use, we record the following estimate

Lemma 2.2.2. *Let n, k be positive integers. Fix m such that $k \le m \le n+k$, let Q be the parabolic subgroup of G_{n+k} with Levi factor $G_{n+k-m} \times (G_1)^m$. For any $x = \begin{pmatrix} x_1 \\ \cdots \\ x_k \end{pmatrix} \in \mathrm{Mat}_{k\times n}(\mathbb{A}), x_i \in \mathbb{A}_n$, assume that under the Iwasawa decomposition $G_{n+k}(\mathbb{A}) = N_Q(\mathbb{A})M_Q(\mathbb{A})K_{n+k}$, we write*

$$\begin{pmatrix} 1_n & \\ x & 1_k \end{pmatrix} = u(x) \begin{pmatrix} g(x) & \\ & t(x) \end{pmatrix} k(x), \tag{2.2.2}$$

where $g(x) \in \mathrm{GL}_{n+k-m}(\mathbb{A})$ *and* $t(x) = \mathrm{diag}(t_1(x), \cdots, t_m(x))$. *Then there exists* $M > 0$ *such that*

(2.2.3) For $1 \leq i \leq k$, *we have* $|t_{m+k-i}(x) \cdots t_m(x)| \gg \|x_i\|_{\mathbb{A}_n}$,
(2.2.4) $\|g(x)\|_{\mathrm{GL}_n(\mathbb{A})} \ll \|x\|^M_{\mathrm{Mat}_{k\times n}(\mathbb{A})}$, $\|t_i(x)\|_{\mathrm{GL}_1(\mathbb{A})} \ll \|x\|^M_{\mathrm{Mat}_{k\times n}(\mathbb{A})}$.

holds for any $x \in \mathrm{Mat}_{k\times n}(\mathbb{A}_n)$ *and* $1 \leq i \leq k$.

Remark 2.2.3. Since different choices of Iwasawa decomposition will yield right translation of $g(x)$ or $t(x)$ by elements of K_{m+k}, hence (2.2.3) is independent of the choice of Iwasawa decomposition and (2.2.4) holds for any choice of Iwasawa decomposition (after possibly enlarging constant).

Proof. (2.2.4) follows from (2.2.1). Now we prove (2.2.3). Let $e_1, \cdots, e_{n+k}$ be the canonical basis for F_{n+k}. The basis e_i yield a canonical basis $\{e_I := \bigwedge_{i\in I} e_i\}_{I\subset\{1,\cdots,k\},|I|=i}$ for the exterior power $\bigwedge^i F_{n+k}$. For $\omega \in \bigwedge^i \mathbb{A}_{n+k}$, write $\omega = \sum_I a_I e_I$, we define

$$|\omega| = \prod_v \max_I \{|a_I|_v\}.$$

For any $g \in G_{n+m}(\mathbb{A})$ and $\omega \in \bigwedge^i \mathbb{A}_{n+k}$, we denote by $\omega \cdot g$ the natural action of g on ω. The absolute value $|\cdot|$ satisfies

$$|\omega| \ll |\omega \cdot k| \ll |\omega|, \quad \forall \omega \in \bigwedge^i \mathbb{A}_{n+k}, k \in K_{n+k}. \tag{2.2.5}$$

Let $1 \leq i \leq k$. Consider $\omega_i := e_{n+i} \wedge \cdots \wedge e_{n+k} \in \bigwedge^{n-i} \mathbb{A}_{n+k}$. Since $\omega_{k-i+1} \cdot u = \omega_i$ for any $u \in N_Q(\mathbb{A})$. We can check that

$$\left| \omega_i \cdot \begin{pmatrix} 1_n & \\ x & 1_k \end{pmatrix} \right| \gg \prod_v \max\{|x_{i,1}|_v, \cdots, |x_{i,n}|_v, 1\} = \|x_i\|_{\mathbb{A}_n}. \tag{2.2.6}$$

By (2.2.5), applying right hand side of (2.2.2) to ω yields

$$\left| \omega_i \cdot \begin{pmatrix} 1_n & \\ x & 1_k \end{pmatrix} \right| \ll |t_{m+k-i}(x) \cdots t_m(x)|. \tag{2.2.7}$$

Combining (2.2.6) and (2.2.7) yields (2.2.3). $\square$

2.2.6. *The map* H_P. We denote by A_G^∞ the neutral component of real points of the maximal split central torus of $\mathrm{Res}_{F/\mathbb{Q}} G$. For a semi-standard parabolic subgroup P of G, let $A_P^\infty := A_{M_P}^\infty$. We also define $A_0^\infty := A_{P_0}^\infty = A_{M_0}^\infty$.

The map

$$H_P : P(\mathbb{A}) \to \mathfrak{a}_P,\, p \mapsto (\chi \mapsto \log|\chi(g)|), \quad \chi \in X^*(P),$$

extends to $G(\mathbb{A})$, by requiring it trivial on K. The map H_P induces an isomorphism $A_P^\infty \cong \mathfrak{a}_P$, we endow A_P^∞ with the Haar measure such that this isomorphism is measure-preserving.

2.2.7. *An estimate.*

Lemma 2.2.4. *For every $k \geq n$, if N is sufficiently large, we have*

$$\sum_{v \in F_n \setminus \{0\}} \|av\|_{\mathbb{A}_n}^{-N} \ll |a|^{-k}, \quad a \in \mathbb{A}^\times.$$

Proof. We write a as $a^1 t$, where $t \in \mathbb{R}_{>0}$ and $|a^1| = 1$. Then

$$\sum_{v \in F_n \setminus \{0\}} \|av\|_{\mathbb{A}_n}^{-N} \ll \sum_{v \in F_n \setminus \{0\}} \|tv\|_{\mathbb{A}_n}^{-N} \|a^1\|_{\mathbb{A}^\times}^{N}$$

Since the LHS is invariant under $F^\times$, we have

$$\sum_{v \in F_n \setminus \{0\}} \|av\|_{\mathbb{A}_n}^{-N} \ll \sum_{v \in F_n \setminus \{0\}} \|tv\|_{\mathbb{A}_n}^{-N} \|a^1\|_{\mathbb{G}_m}^{N}$$

Since $[\mathbb{G}_m]^1$ is compact, $\|a^1\|_{\mathbb{G}_m}^N$ is bounded. Therefore we are reduced to the case when $a \in \mathbb{R}_{>0}$, in which case, it is proved in [BCZ22, (2.6.2.6)]. □

Corollary 2.2.5. *For any $c > 1$, there exists N_0 such that for any $N \geq N_0$, the integral*

$$\int_{\mathbb{A}^\times} \|x\|_{\mathbb{A}}^{-N} |x|^s \mathrm{d}x$$

is absolutely convergent for $1 < \mathrm{Re}(s) < c$.

Proof. We write $[\mathrm{GL}_1]^{\leq 1}$ (resp. $[\mathrm{GL}_1]^{\geq 1}$) for the elements $x \in [\mathrm{GL}_1]$ such that $|x| \leq 1$ (resp. ≥ 1). We write the integral as

$$\int_{[\mathrm{GL}_1]} \sum_{v \neq 0} \|vx\|_{\mathbb{A}}^{-N} |x|^s \mathrm{d}x.$$

By Lemma 2.2.4, when N is sufficiently large, it is essentially bounded by

$$\int_{[\mathrm{GL}_1]^{\leq 1}} |x|^{s-1} \mathrm{d}x + \int_{[\mathrm{GL}_1]^{\geq 1}} |x|^{s-c} \mathrm{d}x.$$

This is finite when $1 < \mathrm{Re}(s) < c$. □

2.3. **Spaces of functions.**

2.3.1. There are various function spaces on $[G]_P$ which we briefly recall below. The reader may consult [BCZ22, §2.5] for more details.

A function $f : G(\mathbb{A}) \to \mathbb{C}$ is called *smooth*, if it is right J-invariant for some open compact subgroup $J \subset G(\mathbb{A}_f)$ and for any $g_f \in G(\mathbb{A}_f)$, the function $g_\infty \mapsto f(g_f g_\infty)$ is C^∞. A function on $[G]_P$ is called smooth if it pulls back to a smooth function on $G(\mathbb{A})$.

Let $\mathcal{S}([G]_P)$ be the space of *Schwartz functions* on $[G]_P$. It is the union of $\mathcal{S}([G]_P, J)$ for open compact subgroup $J \subset G(\mathbb{A}_f)$. Where $\mathcal{S}([G]_P, J)$ is the space of smooth functions on $[G]_P$ which are right J invariant and

$$\|f\|_{X,N} := \sup_{x\in[G]_P} |R(X)f(x)|\|x\|_P^N < \infty$$

for any $X \in \mathcal{U}(\mathfrak{g}_\infty)$ and $N > 0$. The vector space $\mathcal{S}([G]_P, J)$ is naturally a Fréchet space and $\mathcal{S}([G]_P)$ is naturally a strict LF space.

For $N > 0$, let $\mathcal{S}_N([G]_P)$ be the set of smooth functions f on $[G]_P$ such that $\|f\|_{X,N} < \infty$ for all $X \in \mathcal{U}(\mathfrak{g}_\infty)$. It is also a natural LF space.

Let $\mathcal{S}^0([G]_P)$ be the space of measurable function f on $[G]_P$ such that

$$\|f\|_{\infty,N} := \sup_{x\in[G]_P} |f(x)|\|x\|_P^N < \infty \tag{2.3.1}$$

for any $N > 0$. It is naturally a Fréchet space.

Let $\mathcal{T}([G]_P)$ be the function of *uniform moderate growth* on $[G]_P$. It is the union of $\mathcal{T}_N([G]_P, J)$, where $N > 0$ and $J \subset G(\mathbb{A}_f)$ is open compact subgroup. $\mathcal{T}_N([G]_P, J)$ consists of smooth functions f on $[G]_P$ which are right J-invariant and

$$\|f\|_{X,-N} := \sup_{x\in[G]_P} |R(X)f(x)|\|x\|_P^{-N} < \infty$$

for any $X \in \mathcal{U}(\mathfrak{g}_\infty)$. The vector space $\mathcal{T}_N([G]_P, J)$ is naturally a Fréchet space and $\mathcal{T}([G]_P)$ then carries the induced (non-strict) LF topology.

For a Hilbert representation V of $G(\mathbb{A})$, we write V^∞ for the set of smooth vectors, i.e. the set $v \in V$ that is fixed by a compact open subgroup of $G(\mathbb{A}_f)$ and is a smooth vector as $G(F_\infty)$ representation. For each compact open subgroup $J \subset G(\mathbb{A}_f)$, the vector space $V^{\infty,J}$ carries the usual Fréchet topology (for smooth vectors in a Lie group representation). We endow $V^\infty = \bigcup_J V^{\infty,J}$ the LF topology.

For an integer N, we write $L^2_N([G]_P)$ for the weighted L^2 space consisting of measurable functions f on $[G]_P$ such that

$$\int_{[G]_P} |f(x)|^2\|x\|_P^N \mathrm{d}x < \infty.$$

Let $L^2_N([G]_P)^\infty$ be the set of smooth vectors. By Sobolev lemma [Ber88, §3.4, Key Lemma], we have

(2.3.2) For each $N > 0$ there exists $N' > 0$ such that we have closed embedding of topological vector spaces

$$L^2_N([G]_P)^\infty \hookrightarrow S_{N'}([G]_P), \quad \mathcal{S}_N([G]_P) \hookrightarrow L^2_{N'}([G]_P)^\infty.$$

2.3.2. *Constant terms.* For $P \subset Q$, we have the following *constant term* map

$$\mathcal{T}([G]_Q) \ni f \mapsto f_P := \left(g \mapsto \int_{[N_P]} f(ng)\mathrm{d}n \right) \in \mathcal{T}([G]_P).$$

We recall the following useful estimate of constant term of a Schwartz function [BCZ22, Lemma 2.5.13.1]

Lemma 2.3.1. *Let P be a parabolic subgroup of G. Then there is a constant $c > 0$ such that for every $N \geq 0$,*

$$f \mapsto \sup_{x \in [G]_P} \delta_P(x)^{cN} \|x\|_P^N |f_P(x)|$$

is a continuous semi-norm on $\mathcal{S}([G])$.

As a direct consequence, we obtain

(2.3.3) Let P be a standard parabolic subgroup of G and $\kappa \in C_c^\infty(\mathfrak{a}_P)$ be a compactly supported smooth function on $\mathfrak{a}_P$. Then for every $f \in \mathcal{S}([G])$, we have

$$(\kappa \circ H_P) \cdot f_P \in \mathcal{S}([G]_P).$$

Also, combining Lemma 2.3.1 with (2.3.2), we obtain:

(2.3.4) For $N > 0$. There exists $c_N > 0$ such that for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c_N$, we have

$$[M_P] \ni m \mapsto f_P(m)\delta_P(m)^s \in L^2_N([M_P])^\infty.$$

The following two variable versions of the constant term estimates follows from the same proof of [BCZ22, Lemma 2.5.13.1]

Lemma 2.3.2. *Let G, H be connected reductive groups over F. Let $P \times Q$ be a parabolic subgroup of $G \times H$. Then there exists $c > 0$ such that for any $M, N \geq 0$,*

$$f \mapsto \sup_{(x,y) \in [G \times H]_{P \times Q}} \delta_P(x)^{cN} \delta_Q(y)^{cM} \|x\|_P^N \|y\|_Q^M |f_{P \times Q}(x, y)|$$

is a continuous semi-norm on $\mathcal{S}([G \times H])$.

2.3.3. *Polarized Θ-series.* Let $\Phi \in \mathcal{S}(\mathbb{A}_n)$. We associate the following Θ*-series* :

$$\Theta(g, \Phi) = \sum_{v \in F_n} \Phi(vg)|g|^{\frac{1}{2}}, \quad g \in [\mathrm{GL}_n] \tag{2.3.5}$$

The factor $|g|^{\frac{1}{2}}$ appears because the action of $\mathrm{GL}_n(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}_n)$ given by $(g \cdot \Phi)(v) = \Phi(vg)|g|^{\frac{1}{2}}$ is unitary.

The convergence and the growth of the Θ-series are justified by the following lemma

Lemma 2.3.3. *There exists $M > 0$ and $N_0 > 0$, such that for every $N \geq N_0$, we have*

$$\sum_{v \in F_n} \|vh\|_{\mathbb{A}_n}^{-N} \ll \|h\|_{G_n}^{M}. \tag{2.3.6}$$

In particular, there exists $N_0 > 0$ such that for any $\Phi \in \mathcal{S}(\mathbb{A}_n)$, we have $\Theta(\cdot, \Phi) \in \mathcal{T}_M([G_n])$.

Proof. Note that the left-hand side of (2.3.6) is decreasing in N, so it suffices to find $N = N_0$ such that (2.3.6) holds. There exists $c > 0$ such that

$$\|vh\|_{\mathbb{A}_n}^{-N} \ll \|v\|_{\mathbb{A}_n}^{-cN} \|h\|_{G_n(\mathbb{A})}^{cN}$$

holds for any $v \in \mathbb{A}_n$ and $h \in G_n(\mathbb{A})$. It then suffices to pick $N_0 > 0$ such that $\sum_{v \in F_n} \|v\|_{\mathbb{A}_n}^{-cN_0} < \infty$. □

Corollary 2.3.4. *For any $C > 0$, there exists $N_0 > 0$, such that for any $N, N' > N_0$, the integral*

$$\int_{\mathcal{P}_n(F)\backslash G_n(\mathbb{A})} \|e_n h\|_{\mathbb{A}_n}^{-N} |\det h|^s \|h\|_{G_n}^{-N'} \mathrm{d}h$$

converges for $|\mathrm{Re}(s)| < C$.

Proof. The integral can be written as

$$\int_{[G_n]} \sum_{v \neq 0} \|vh\|_{\mathbb{A}_n}^{-N} |\det h|^s \|h\|_{G_n}^{-N'} \mathrm{d}h.$$

By Lemma 2.3.3 and the fact that

$$\max\{|\det h|, |\det h|^{-1}\} \ll \|h\|_{G_n}^{r}$$

for some $r > 0$, we see that the integral is bounded by

$$\int_{[G_n]} \|h\|_{G_n}^{-N'+M+\mathrm{Re}(s)r} \mathrm{d}h,$$

for some $M > 0$, the result follows. □

We also remark that, by the Poisson summation formula, Θ-series satisfies

$$\Theta(g, \Phi) = \Theta({}^t g^{-1}, \widehat{\Phi}). \tag{2.3.7}$$

2.3.4. *Estimates on Fourier coefficients.* Let $P \subset G$ be a standard parabolic subgroup, $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be a non-trivial character and V_P be the vector space of additive algebraic characters $N_P \to \mathbb{G}_a$. Let $l \in V_P(F)$ and set $\psi_l := \psi \circ l_{\mathbb{A}} : [N_P] \to \mathbb{C}^\times$ where $l_{\mathbb{A}}$ denotes the homomorphism between adelic points $N_P(\mathbb{A}) \to \mathbb{A}$. For $\varphi \in \mathcal{T}([G])$, we set

$$\varphi_{N_P, \psi_l}(g) = \int_{[N_P]} \varphi(ug) \psi_l(u)^{-1} \mathrm{d}u, \quad g \in G(\mathbb{A}).$$

The adjoint action of M_P on N_P induces one on V_P that we denote by Ad^*.

Lemma 2.3.5. *[BCZ22, Lemma 2.6.1.1]*

(1) There exists $c > 0$ such that for every $N_1, N_2 \geq 0$,

$$\varphi \mapsto \sup_{m \in M_P(\mathbb{A})} \sup_{k \in K} \| \operatorname{Ad}^*(m^{-1})l\|_{V_P(\mathbb{A})}^{N_1} \|m\|_{M_P}^{N_2} \delta_P(m)^{cN_2} |\varphi_{N_P, \psi_l}(mk)|$$

is a continuous semi-norm on $\mathcal{S}([G])$.

(2) Let $N > 0$. Then, for every $N_1 \geq 0$,

$$\varphi \mapsto \sup_{m \in M_P(\mathbb{A})} \sup_{k \in K} \| \operatorname{Ad}^*(m^{-1})l\|_{V_P(\mathbb{A})}^{N_1} \|m\|_{M_P}^{-N} |\varphi_{N_P, \psi_l}(mk)|$$

is a continuous semi-norm on $\mathcal{T}_N([G])$.

Proof. Without the term $\sup_{k \in K}$, this is exactly [BCZ22, Lemma 2.6.1.1]. Since for any continuous semi-norm $\| \cdot \|$ on $\mathcal{S}([G])$ or $\mathcal{T}_N(G)$, $f \mapsto \sup_{k \in K} \|R(k)f\|$ is still a continuous semi-norm. The result follows. $\square$

2.4. **Automorphic forms and Eisenstein series.**

2.4.1. *Automorphic forms.* Let G be a connected reductive group over F and let P be a standard parabolic subgroup. An *automorphic form* on $[G]_P$ is, by definition, is a $Z(\mathfrak{g}_\infty)$-finite function in $\mathcal{T}([G]_P)$. We denote by $\mathcal{A}_P(G)$ the set of automorphic form on $[G]_P$.

Let $\mathcal{A}_{P,\mathrm{cusp}}(G)$ denote the subspace of $\mathcal{A}_P(G)$ consisting of cuspidal automorphic forms, that is, consisting of $\varphi \in \mathcal{A}_P(G)$ such that $\varphi_Q = 0$ for any standard parabolic subgroups $Q \subset P$.

Let $\mathcal{J}$ be a finite codimensional ideal of $\mathcal{Z}(\mathfrak{g}_\infty)$. Let $\mathcal{A}_{P,\mathcal{J}}(G)$ denote the subspace of automorphic form $\varphi \in \mathcal{A}_P(G)$ such that $R(z)\varphi = 0$ for all $z \in \mathcal{J}$. Then there exists $N > 0$ such that $\mathcal{A}_{P,J}(G)$ is a closed subspace of $\mathcal{T}_N([G])$. We endow $\mathcal{A}_{P,J}$ with the topology induced from $\mathcal{T}_N(G)$. This topology is independent of the choice of N. We then endow $\mathcal{A}_P(G)$ with the inductive limit topology $\mathcal{A}_P(G) = \bigcup_{\mathcal{J}} \mathcal{A}_{P,\mathcal{J}}(G)$. For each $\mathcal{J}$, the inclusion $\mathcal{A}_{P,\mathcal{J}}(G) \hookrightarrow \mathcal{A}_P(G)$ is a closed embedding. We refer the reader to [BCZ22, §2.7.1] for the proof of these facts.

A *cuspidal automorphic representation* of G is defined to be a topologically irreducible subrepresentation π of $G(\mathbb{A})$ on $\mathcal{A}_{\mathrm{cusp}}(G)$. Note that a cuspidal automorphic representation π is unitary if and only if any $\varphi \in \pi$ has a unitary central character, in the sense that for any $z \in G(\mathbb{A})$, $\varphi(zg) = \varphi(g)\omega(z)$ for some unitary character $\omega : Z_G(\mathbb{A}) \to \mathbb{C}^\times$, where Z_G denote the center of G.

2.4.2. *Eisenstein series.* Let $P = M_P N_P$ be a standard parabolic subgroup of G. Let π be a cuspidal automorphic representation of M_P. Let $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$ denote the the sum of all cuspidal automorphic representations of $M_P(\mathbb{A})$ that are isomorphic to π.

We write $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$ (resp. $\mathcal{A}_{P,\pi}$) for the subspace

$$\{\varphi \in \mathcal{A}_P(G) \mid \text{ for any } g \in G(\mathbb{A}), m \mapsto \delta_P^{-\frac{1}{2}}(m)\varphi(mg) \in \pi \text{ (resp. } \mathcal{A}_{\pi,\mathrm{cusp}}(M_P))\}.$$

of $\mathcal{A}_P(G)$.

For $\varphi \in \mathcal{A}_{P,\pi}$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we define the *Eisenstein series* as

$$E(g, \varphi, \lambda) = \sum_{\gamma \in P(F)\backslash G(F)} e^{\langle \lambda, H_P(\gamma g)\rangle} \varphi(\gamma g).$$

The series is absolutely convergent when $\mathrm{Re}(\lambda)$ lies in some cones and by [BL24], [Lap08], it has meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^*$. When π is unitary, for $\varphi \in \mathcal{A}_{P,\pi}$, the Eisenstein series $E(g, \varphi, \lambda)$ is regular when $\lambda \in i\mathfrak{a}_P^*$.

2.4.3. *Intertwining operators and normalizations.* Let P and Q be standard parabolic subgroups of G. For any $w \in W(P, Q)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, the *intertwining operator*

$$M(w, \lambda) : \mathcal{A}_P(G) \to \mathcal{A}_Q(G)$$

is defined by the meromorphic continuation of the integral

$$\begin{aligned}(M(w,\lambda)\varphi)(g) = {} & \exp(-\langle w\lambda, H_P(g)\rangle) \\ & \times \int_{(N_Q \cap wN_Pw^{-1})(\mathbb{A})\backslash N_Q(\mathbb{A})} \exp(\langle \lambda, H_P(w^{-1}ng)\rangle)\varphi(w^{-1}ng)\mathrm{d}n.\end{aligned}$$

(see [BL24] for the meromorphic continuation). Let π be a cuspidal representation of M_P, we denote by $M_\pi(w, \lambda)$ the restriction of $M(w, \lambda)$ to the subspace $\mathcal{A}_{P,\pi}(G) \subset \mathcal{A}_P(G)$. It is known that if π is unitary, then $M_\pi(w, \lambda)$ is regular on $i\mathfrak{a}_P^*$.

Now we assume G is G_n and write $M_P = G_{n_1} \times \cdots \times G_{n_k}$ and $\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$. Let $\Sigma_P^+ \subset X^*(A_P)$ denote the set of positive roots of A_P action on $\mathfrak{n}_P$. Let $\beta \in \Sigma_P^+$ be the positive root of P associated to the two blocks G_{n_i} and G_{n_j} with $1 \leq i < j \leq k$. Set

$$n_\pi(\beta, s) = \frac{L(s, \pi_i \times \pi_j^\vee)}{\epsilon(s, \pi_i \times \pi_j^\vee)L(1+s, \pi_i \times \pi_j^\vee)} = \frac{L(1-s, \pi_i^\vee \times \pi_j)}{L(1+s, \pi_i \times \pi_j^\vee)},$$

then we define

$$n_\pi(w, \lambda), = \prod_{\substack{\beta \in \Sigma_P \\ w\beta < 0}} n_\pi(\beta, \langle \lambda, \beta^\vee \rangle).$$

Following [MW89], we normalize $M(w, \lambda)$ as

$$M_\pi(w, \lambda) = n_\pi(w, \lambda) N_\pi(w, \lambda). \tag{2.4.1}$$

Let $\varphi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$. Assume that $\varphi = \otimes_v' \varphi_v$ is factorizable, where $\varphi_v \in \mathrm{Ind}_{P(F_v)}^{G(F_v)} \pi_v$. Let $\mathtt{S}$ be a sufficiently large finite set of places of F, which we assume to contain Archimedean places as well as the places φ_v is ramified. We have a factorization

$$N_\pi(w, \lambda)\varphi = \prod_{v \in \mathtt{S}} N_{\pi_v}(w, \lambda)\varphi_v. \tag{2.4.2}$$

Here $N_{\pi_v}(w, \lambda)$ is the meromorphic *local normalized intertwining operator* $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \pi_{v,\lambda} \to \mathrm{Ind}_{Q(F_v)}^{G(F_v)} (w\pi)_{v,\lambda}$, (see [MW89]). The product notation of (2.4.2) means $N_\pi(w, \lambda)\varphi$ is factorizable and for $v \notin \mathtt{S}$ the local component $N_{\pi_v}(w, \lambda)\varphi_v$ is the unique unramified vector in

$\mathrm{Ind}_{Q(F_v)}^{G(F_v)}(w\pi)_v$ such that $N_{\pi_v}(w,\lambda)\varphi_v(1)$ corresponds to $\varphi_v(1)$ under the natural identification between π_v and $(w\pi)_v$.

The following result is taken from [MW89, Page 607]

Lemma 2.4.1. *Let π_v be a smooth irreducible and unitary representation of $M_P(F_v)$. Then the operator $N_{\pi_v}(w,\lambda)$ is holomorphic and unitary if $\lambda \in i\mathfrak{a}_P^*$. It is an isomorphism.*

From now on, we simply write $N_\pi(w)$ for $N_\pi(w,0)$. If we put

$$L(s,\pi,\widehat{\mathfrak{n}}_P^-) := \prod_{1\le i<j\le k} L(s,\pi_i \times \pi_j^\vee), \tag{2.4.3}$$

then we have

$$n_\pi(w,0) = \frac{L(1,w\pi,\widehat{\mathfrak{n}}_Q^-)}{L(1,\pi,\widehat{\mathfrak{n}}_P^-)}.$$

For sufficiently large $\mathtt{S}$ as above, we denote by

$$N_{\pi,\mathtt{S}}(w) = \prod_{v\in\mathtt{S}} N_{\pi_v}(w) : \mathrm{Ind}_{P(F_\mathtt{S})}^{G(F_\mathtt{S})}\pi_\mathtt{S} \to \mathrm{Ind}_{Q(F_\mathtt{S})}^{G(F_\mathtt{S})}(w\pi)_\mathtt{S}.$$

We finally remark that the normalized intertwining operator naturally extends to the case when G is a product of G_{n_i}.

2.5. **Langlands spectral decomposition.**

2.5.1. *Cuspidal datum.* Let G be a connected reductive group over F. Let $\underline{\mathfrak{X}}(G)$ denote the set of pairs (M_P,π), where M_P is the Levi component of a standard parabolic subgroup P and π is a cuspidal automorphic representation of $M_P(\mathbb{A})$ with central character trivial on A_P^∞. Two elements (M_P,π) and (M_Q,π') of $\underline{\mathfrak{X}}(G)$ are called equivalent, if there exists $g\in G(F)$ such that $gM_Pg^{-1}=M_Q$ and $g\pi=\pi'$. Let $\mathfrak{X}(G)$ denote the equivalence class of $\underline{\mathfrak{X}}(G)$, an element of $\mathfrak{X}(G)$ will be called a *cuspidal data*.

For a standard parabolic subgroup $P\subset G$, there exists a natural map $\underline{\mathfrak{X}}(M_P)\to\underline{\mathfrak{X}}(G)$, and it induces a map $\mathfrak{X}(M_P)\to\mathfrak{X}(G)$ which has finite fiber. For each subset $\mathfrak{X}\subset\mathfrak{X}(G)$, we will write $\mathfrak{X}^M$ for its preimage in $\mathfrak{X}(M_P)$.

2.5.2. *Langlands decomposition.* For $\chi\in\underline{\mathfrak{X}}(G)$, and P be a standard parabolic subgroup, we write $\mathfrak{D}_\chi^P\subset\mathcal{S}([G]_P)$ the set of *pseudo-Eisenstein series* with respect to χ (See [MW95, §II.1], [BCZ22, §2.9]). Let $L_\chi^2([G]_P)$ denote the closure of $\mathfrak{D}_\chi^P$ in $L^2([G]_P)$, then we have the following coarse Langlands decomposition:

$$L^2([G]_P) = \bigoplus_{\chi\in\mathfrak{X}(G)} L_\chi^2([G]_P). \tag{2.5.1}$$

For a subset $\mathfrak{X}\subset\mathfrak{X}(G)$, we write $L_\mathfrak{X}^2([G]_P) := \widehat{\bigoplus}_{\chi\in\mathfrak{X}} L_\chi^2([G]_P)$. Then we define

$$\mathcal{S}_\mathfrak{X}([G]_P) := L_\mathfrak{X}^2([G]_P)\cap\mathcal{S}([G]_P). \tag{2.5.2}$$

Note that $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is a closed subspace of $\mathcal{S}([G]_P)$, since it is orthogonal complement of $\bigcup_{\chi\notin\mathfrak{X}}\mathfrak{O}_\chi$ in $\mathcal{S}([G]_P)$.

We then define $\mathcal{T}_{\mathfrak{X}}([G]_P)$ (resp. $L^2_N([G]_P)^\infty$) be the orthogonal complement of $\mathcal{S}_{\mathfrak{X}^c}([G]_P)$ in $\mathcal{T}([G]_P)$ (resp. $L^2_N([G]_P)^\infty$).

We call element of $\mathcal{S}_{\mathfrak{X}}([G]_P)$ the set of *Schwartz function with cuspidal support in* $\mathfrak{X}$ and $\mathcal{T}_{\mathfrak{X}}([G]_P)$ the set of *uniform moderate growth function with cuspidal support in* $\mathfrak{X}$. For any subset $\mathfrak{X}\subset\mathfrak{X}(G)$, the space $\mathcal{S}_{\mathfrak{X}}([G]_P)$ is dense in $\mathcal{T}_{\mathfrak{X}}([G]_P)$ (see [BCZ22, §2.9.5])

The following theorem [BCZ22, Theorem 2.9.4.1] describes the decomposition of a function according to cuspidal support:

Theorem 2.5.1 (Beuzart-Plessis-Chaudouard-Zydor)**.** *We have the following statements:*

(1) For $f\in\mathcal{S}([G]_P)$, let f_χ denote the χ-part of f under the decomposition (2.5.1)*, then $f_\chi\in\mathcal{S}([G]_P)$ and $f=\sum_\chi f_\chi$, where the sum is absolutely summable in $\mathcal{S}([G]_P)$.*

(2) The map $f\mapsto f_\chi:\mathcal{S}([G]_P)\to\mathcal{T}([G]_P)$ extends by continuity to a map $\mathcal{T}([G]_P)\to\mathcal{T}([G]_P)$, which we still denote by $f\mapsto f_\chi$. Then for any $f\in\mathcal{T}([G]_P)$, $f_\chi\in\mathcal{T}_\chi([G]_P)$ and the sum $f=\sum_\chi f_\chi$ is absolutely summable in $\mathcal{T}([G]_P)$.

2.5.3. *Some lemmas.*

Lemma 2.5.2. *For each $\chi\in\mathfrak{X}(G)$, $\mathfrak{O}^P_\chi$ is dense in $\mathcal{S}_\chi([G]_P)$ and $\mathcal{T}_\chi([G]_P)$.*

Proof. See [Boi25, Lemma 5.5.1.2] for the density in $\mathcal{S}_\chi([G]_P)$, the density in $\mathcal{T}_\chi([G]_P)$ also follows, since $\mathcal{S}_\chi([G]_P)$ is dense in $\mathcal{T}_\chi([G]_P)$. □

Lemma 2.5.3. *Let $\chi\in\mathfrak{X}(G)$ be a cuspidal datum and P be a standard parabolic subgroup of G. Then we have*

$$E_P^G(\mathcal{S}_\chi([G]_P))\subset\mathcal{S}_\chi([G]),\quad \mathcal{T}_\chi([G])_P\subset\mathcal{T}_\chi([G]_P).$$

Proof. See [BCZ22, Lemma 2.9.3.1]. □

Lemma 2.5.4. *Let $\chi\in\mathfrak{X}(G)$ be a cuspidal datum, P be a standard parabolic subgroup of G and χ_M be the inverse image of χ in $\mathfrak{X}(M_P)$. Then, for every $f\in\mathcal{S}_\chi([G]_P)$(resp. $\mathcal{T}_\chi([G]_P)$), its restriction $f|_{[M_P]}$ to $[M_P]$ belongs to $\mathcal{S}_{\chi^M}([M_P])$(resp. $\mathcal{T}_{\chi^M}([M_P])$).*

Proof. If $f\in\mathfrak{O}^P_\chi$, this follows from the definition, and the general cases follow by the density (Lemma 2.5.2). □

Lemma 2.5.5. *Assume that $G=H\times L$, where H and L are connected reductive groups over F. Then we have a natural identification $\mathfrak{X}(G)=\mathfrak{X}(H)\times\mathfrak{X}(L)$. For a subset $\mathfrak{X}\subset\mathfrak{X}(G)$, denote its projection to $\mathfrak{X}(H)$ by $\mathfrak{X}_H$. Then for every $\mathcal{S}_{\mathfrak{X}}([G])$, its restriction to $[H]$ belongs to $\mathcal{S}_{\mathfrak{X}_H}([H])$.*

The proof is the same as the proof of Lemma 2.5.4.

Lemma 2.5.6. *Let $\chi \in \mathfrak{X}(G)$ be a cuspidal datum, P be a standard parabolic subgroup of G, and $\kappa \in C_c^\infty(\mathfrak{a}_P)$ be a compactly supported smooth function on $\mathfrak{a}_P$. Then $\mathcal{T}_\chi([G]_P)$ is stable under the multiplication by $\kappa \circ H_P$.*

Proof. This can also be proved via the method in the proof of Lemma 2.5.4. Alternatively, for $f \in \mathcal{T}_\chi([G]_P)$, we need to show that $f \cdot (\kappa \circ H_P)$ is orthogonal to any $f' \in \mathcal{S}_{\chi'}([G]_P)$ for $\chi' \neq \chi$. Then

$$\langle (\kappa \circ H_P) f, f' \rangle_{[G]_P} = \langle (f, (\kappa \circ H_P) f' \rangle_{[G]_P}.$$

Therefore, it reduces to proving $\mathcal{S}_\chi$ is stable under multiplication by $\kappa \circ H_P$. Since $\mathcal{S}_\chi$ is orthogonal to $\mathfrak{O}_{\chi'}$ to all $\chi' \neq \chi$. By the same trick, it reduces to proving each $\mathfrak{O}_\chi$ is stable under multiplication by $(\kappa \circ H_P)$, which follows from the definition. $\square$

2.6. **Whittaker model.**

2.6.1. *Local Whittaker model.* We now assume that F be a local field. Let G be a quasi-split group over F. We fix a splitting $\mathrm{Spl} = (B, T, \{X_\alpha\}_\alpha)$ of G. This means $B = TU$ is an F-Borel subgroup, T is a maximal torus and $\{X_\alpha\}_\alpha$ is a set of Γ_F invariant root vector.

We fix a splitting Spl of G and an additive character $\psi : F \to \mathbb{C}^\times$. They give rise to a Whittaker data $\mathfrak{w} = \mathfrak{w}_{(\mathrm{Spl},\psi)} = (B, \psi_U)$ of G. More generally, for any Levi subgroup M containing T, they give rise to a Whittaker data $\mathfrak{w}_M = \mathfrak{w}_{M,\mathrm{Spl},\psi} = (B_M, \psi_{U_M})$ of M, where $B_M := B \cap M$ and $\psi_{U_M} : U_M := U \cap M \to \mathbb{C}^\times$ is the character induced by Spl and ψ.

Let π be an irreducible representation of $G(F)$. Recall that π is called *generic*, if it satisfies $\mathrm{Hom}_{U(F)}(\pi, \psi_U) \neq 0$. When π is generic, it can be identified with its *Whittaker model* $\mathcal{W}(\pi, \psi_U)$. Recall that

$$\mathcal{W}(\pi, \psi_U) = \{g \mapsto \lambda(\pi(g)v) \mid v \in \pi\} \subset C^\infty(U(F)\backslash G(F), \psi_U),$$

where λ is any non-zero element of $\mathrm{Hom}_{U(F)}(\pi, \psi_U) \neq 0$.

When $G = G_{n_1} \times \cdots \times G_{n_k}$ is a product of general linear groups, we always fix the standard splitting.

2.6.2. *Jacquet integral.* Let $P = M_P N_P$ be a parabolic subgroup and let π be an irreducible generic representation of $M_P(F)$.

(2.6.1) $\mathrm{Ind}_{P(F)}^{G(F)} \pi$ can be identified with the following space of functions on $G(F)$.

$$\left\{ W^{M_P} : G(F) \to \mathbb{C} \mid \forall g \in G(F), m \in M(F) \mapsto \delta_P^{-\frac{1}{2}}(m) W^M(mg) \in \mathcal{W}(\pi, \psi_{U_M}) \right\}.$$

We denote this space by $\mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{W}(\pi, \psi_{U_M}))$

Let $N_{\overline{P}}$ be the unipotent radical of the parabolic subgroup $\overline{P}$ of G opposite to P. Let $w_0 = w_\ell w_\ell^P$, where w_ℓ and w_ℓ^M are the longest elements in W and W^P, respectively. Denote by $\widetilde{w_0} \in G(F)$ the *Tits lifting* [LS87, p. 228] of w_0 and let $N' = \widetilde{w_0} N_{\overline{P}} \widetilde{w_0}^{-1}$. The *Jacquet functional* is given by the holomorphic continuation of the Jacquet integral

$$\Omega_\lambda(W^{M_P})(g) = \int_{N'(F)} W^{M_P}(\widetilde{w_0}^{-1} n' g) \psi_U(n')^{-1} \mathrm{d}n'.$$

It induces an isomorphism between $\mathrm{Ind}_{P(F)}^{G(F)} \mathcal{W}(\pi_\lambda, \psi_{U_M})$ and $\mathcal{W}(\mathrm{Ind}_{P(F)}^{G(F)} \pi_\lambda, \psi_U)$, where $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, and π_λ denotes the unramified twist of π by λ. When $\lambda = 0$, we may simply denote Ω_λ by Ω.

More generally, let $P \subset Q$ be standard parabolics. Let $w_0^Q := w_\ell^Q w_\ell^P$. Let $N' = \widetilde{w_0^Q} N_{\overline{P}} \widetilde{w_0^Q}^{-1} \cap M_Q$. Then we have a Jacquet functional Ω^Q from $\mathrm{Ind}_{P(F)}^{G(F)} \mathcal{W}(\pi, \psi_{U_{M_P}}) \to \mathrm{Ind}_{Q(F)}^{G(F)} \mathcal{W}(\mathrm{Ind}_P^Q \pi, \psi_{U_{M_Q}})$, defined by the meromorphic continuation of

$$\Omega_\lambda^Q(W^{M_P})(g) = \int_{N'(F)} W^{M_P}(\widetilde{w_0}^{-1} n' g) \psi_{U_{M_Q}}(n')^{-1} \mathrm{d}n'.$$

2.6.3. Let G be a product of G_{n_i}. Let P, Q be standard parabolics G, and $w \in W(P, Q)$. Let π be a generic representation of $M_P(F)$. The normalized intertwining operator $N(w, \lambda)$: $\mathrm{Ind}_{P(F)}^{G(F)} \pi_\lambda \to \mathrm{Ind}_{Q(F)}^{G(F)} (w\pi)_{w\lambda}$ transports to a map $\mathrm{Ind}_{P(F)}^{G(F)} \mathcal{W}(\pi_\lambda, \psi) \to \mathrm{Ind}_{Q(F)}^{G(F)} \mathcal{W}(w\pi_{w\lambda}, \psi)$, which we will still denote it by $N(w, \lambda)$, and we write $N(w)$ for $N(w, 0)$.

2.6.4. Now let F be a number field. Let N_n be the unipotent radical of the Borel subgroup of G_n, we define a generic character ψ_{N_n} of $[N_n]$ by

$$\psi_{N_n}(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

Assume $G = G_{n_1} \times \cdots \times G_{n_k}$. Let N be the unipotent radical of the Borel subgroup of G and $\psi_N = \psi_{N_{n_1}} \boxtimes \cdots \boxtimes \psi_{N_{n_k}}$ be the generic character on $[N] = [N_{n_1}] \times \cdots \times [N_{n_k}]$. For every $f \in \mathcal{T}([G])$, we set

$$W_f = \int_{[N]} f(ug) \psi_N(u)^{-1} \mathrm{d}u.$$

Let π be a cuspidal representation of $G(\mathbb{A})$, then the map $f \mapsto W_f$ gives an isomorphism between π and its *Whittaker model*

$$\mathcal{W}(\pi, \psi_N) = \{W_f \mid f \in \pi\}.$$

More generally, let P be a standard parabolic of G and $\varphi \in \mathcal{T}([G]_P)$, we set

$$W_\varphi^{M_P}(g) = \int_{[M_P \cap N]} \varphi(ug) \psi_N(u)^{-1} \mathrm{d}u,$$

Let π be a cuspidal unitary representation of $M_P(\mathbb{A})$, then the map $\varphi \in \mathcal{A}_{P,\pi} \mapsto W_\varphi^{M_P}$ gives an isomorphism between $\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$ and the induction of the Whittaker model

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \mathcal{W}(\pi, \psi_N) = \{W^{M_P} : G(\mathbb{A}) \to \mathbb{C}, \forall g \in G(\mathbb{A}), m \in M_P(\mathbb{A}) \mapsto \delta_{P(\mathbb{A})}^{-\frac{1}{2}}(m) W^{M_P}(mg) \in \mathcal{W}(\pi, \psi_N)\}$$

For $f \in \mathcal{T}([G])$ (resp. $\varphi \in \mathcal{T}([G]_P)$) and for a finite set of places $\mathtt{S}$ of F, let $W_{f,\mathtt{S}}$ (resp. $W_{\varphi,\mathtt{S}}^{M_P}$) be the restriction of W_f(resp. $W_\varphi^{M_P}$) to $G(F_{\mathtt{S}})$.

For $\varphi \in \Pi$, write $E(\varphi)(g) = E(g, \varphi, 0)$ for the Eisenstein series of φ. Let $\mathtt{S}$ be a sufficiently large finite set of places of F. Then it follows from [Sha81, §4] that

$$W_{E(\varphi),\mathtt{S}} = L(1, \pi, \widehat{\mathfrak{n}}_P^-)^{-1} \Omega_{\mathtt{S}}(W_{\varphi,\mathtt{S}}^{M_P}), \tag{2.6.2}$$

when $L(1, \pi, \widehat{\mathfrak{n}}_P^-)$ has a pole at $s = 1$, the right hand side is interpreted as 0.

More generally, let R be a standard parabolic subgroup of G containing P, we have that

$$W_{E^R(\varphi),\mathtt{S}}^{M_R} = L(1, \pi, \widehat{\mathfrak{n}_P^R}^-)^{-1} \Omega_{\mathtt{S}}^R(W_{\varphi,\mathtt{S}}^{M_P}). \tag{2.6.3}$$

2.6.5. We still assume that G is a product of G_{n_i}. Let $P = MN$ be a standard parabolic of G and let $\varphi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi = \mathcal{A}_{P,\pi}(G)$ and $\mathtt{S}$ be a sufficiently large finite set of places of F, which we assume to contain Archimedean places as well as places where φ is ramified. Then we have a decomposition $W_\varphi^M = W_{\varphi,\mathtt{S}}^M W_\varphi^{M,\mathtt{S}}$ such that $W_\varphi^{M,\mathtt{S}}(1) = 1$ and is fixed by $K^{\mathtt{S}}$. For $N_\pi(w)\varphi \in \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} w\pi = \mathcal{A}_{Q,w\pi}(G)$, we also have a decomposition

$$W^M(N_\pi(w)\varphi) = W_{\mathtt{S}}^M(N_\pi(w)\varphi) W^{M,\mathtt{S}}(N_\pi(w)\varphi)$$

such that $W^{M,\mathtt{S}}(N_\pi(w)\varphi)(1) = 1$ and is fixed by $K^{\mathtt{S}}$. Then it follows from (2.4.1) and (2.4.2) that

$$W_{\mathtt{S}}^M(N_\pi(w)\varphi) = N_{\pi,\mathtt{S}}(w)(W_{\varphi,\mathtt{S}}^M). \tag{2.6.4}$$

2.7. **Topological vector spaces.** In this article, a LVTVS means a Hausdorff, locally convex topological vector space. We refer the readers to [BCZ22, Appendix A] for more details. Let V, W be two LCTVS. We endow $\mathrm{Hom}(V, W)$ with the pointwise convergence topology. If W is quasi-complete, then so is $\mathrm{Hom}(V, W)$.

Let V, W, X be LCTVS. Let $\mathrm{Bil}_s(V, W; X)$ denote the set of separately bilinear map $V \times W \to X$. It consists of bilinear maps $f : V \times W \to X$ such that for any $v \in V$, the map $f(v, \cdot) : W \to X$ is continuous and for any $w \in W$, the map $f(\cdot, w) : V \to X$ is continuous.

The set $\mathrm{Bil}_s(V, W; X)$ is naturally identified with either $\mathrm{Hom}(V, \mathrm{Hom}(W, X))$ or $\mathrm{Hom}(W, \mathrm{Hom}(V, X))$. Using the weak topology between Hom between any LCTVS, both $\mathrm{Hom}(V, \mathrm{Hom}(W, X))$ and $\mathrm{Hom}(W, \mathrm{Hom}(V, X))$ carry a natural topology. They indeed induce the same topology on $\mathrm{Bil}_s(V, W; X)$, which is in fact the locally convex Hausdorff topology given by the semi-norms $f \mapsto p(f(v, w))$, where (v, w) runs through $V \times W$ and p runs through the continuous semi-norms on X.

The following fact is standard (see e.g. [BL24, §3.2]):

(2.7.1) A map $\mathbb{C} \to \mathrm{Hom}(V, W), s \mapsto T_s$ is holomorphic if and only if for any $v \in V$, then map $\mathbb{C} \ni s \mapsto T_s(v) \in W$ is holomorphic.

Lemma 2.7.1. *(1) Assume that V is LF, W is quasi-complete and let X be a topological space. Let $s \in M \mapsto T_s \in \mathrm{Hom}(V, W)$ be holomorphic and $(s, x) \in M \times X \mapsto v_{s,x} \in V$ be a continuous map which is holomorphic in the first variable. Then, the map $(s, x) \in M \times X \mapsto T_s(v_{s,x}) \in W$ is continuous and holomorphic in the first variable.*

(2) Assume that V and W are LF. Let $s \in M \mapsto B_s \in \mathrm{Bil}_s(V, W)$ be holomorphic and $(s, k) \in M \times K \mapsto v_{s,x} \in V$, $(s, x) \in (M, X) \mapsto w_{s,x} \in W$ be continuous maps which are holomorphic in the first variable. Then, the function $(s, x) \in M \times X \mapsto B_s(v_{s,x}, w_{s,x})$ is continuous and holomorphic in the first variable.

Proof. See [BCZ22, p. 329]. □

The following lemma is standard

Lemma 2.7.2. *Let K be a compact Hausdorff topological group, X be a topological space, and let $f : \mathbb{C} \times K \times X \to \mathbb{C}$ be a continuous map which is holomorphic in the first variable. Then for any $x \in X$*

$$s \in \mathbb{C} \mapsto \int_K f(s, k, x)\mathrm{d}k$$

is holomorphic and the map

$$x \in X \mapsto \int_K f(\cdot, k, x)\mathrm{d}k \in \mathcal{O}(\mathbb{C})$$

is continuous.

Let M be a complex manifold and let V be a topological vector space. A map $f : M \to V$ is said to be *holomorphic*, if for any $\lambda \in V'$, the map $M \ni m \mapsto \langle \lambda, f(m) \rangle$ is holomorphic.

Let $C \in \mathbb{R} \cup \{-\infty\}$ and $f : \mathcal{H}_{>C} \to V$ be a holomorphic map. We say f is *of order at most d in vertical strips* if for every $d' > d$, the function $z \mapsto e^{-|z|^{d'}]} f(z)$ is bounded in vertical strips.

We also recall the following version Phragmen-Lindelöf principle [BCZ22, Corollary A.0.11.2].

Proposition 2.7.3. *Let W be a LF space, and $S \subset W$ be a dense subspace. Let $C > 0$ and $Z_+, Z_- : \mathcal{H}_{>C} \times W \to \mathbb{C}$ be two functions. Assume that*

(1) For every $s \in \mathcal{H}_{>C}, Z_+(s, \cdot)$ and $Z_-(s, \dot{)}$ are continuous functional on W;

(2) There exists $d > 0$ such that for every $w \in W$ and $\epsilon \in \{\pm\}$, $\mathcal{H}_{>C} \ni Z_\varepsilon(s, w)$ is a holomorphic function of order at most d in vertical strips;

(3) For any $f \in S$, $s \mapsto Z_\varepsilon(s, f)$ extends to a holomorphic function on $\mathbb{C}$ of finite order in vertical strips satisfying

$$Z_+(s, f) = Z_-(-s, f).$$

Then Z_+ and Z_- extend to holomorphic functions $\mathbb{C} \to W'$ of finite order in vertical strips satisfying $Z_+(s, w) = Z_-(-s, w)$ for every $(s, w) \in \mathbb{C} \times W$.

3. Canonical extensions of Rankin-Selberg periods – corank 0 and 1

3.1. **Rankin-Selberg period on $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$.** In §3.1, we discuss some results on canonical extension Rankin-Selberg period based on [BCZ22, §7].

3.1.1. *Set up.* Throughout §3.1, let $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$ and let $H = \mathrm{GL}_n$, regarded as the diagonal subgroup $(h, \begin{pmatrix} h & \\ & 1 \end{pmatrix})$ of G. For $f \in \mathcal{S}([G])$, the *Rankin-Selberg period* of f is defined by the absolute convergent integral

$$\mathcal{P}_{\mathrm{RS}}(f) := \int_{[H]} f(h)\mathrm{d}h.$$

3.1.2. *Rankin-Selberg regular cuspidal datum.* Let $\chi \in \mathfrak{X}(G)$ be a cuspidal datum of G. Assume that χ is represented by (M_P, π), and we write

$$M_P = M_{P_n} \times M_{P_{n+1}}, \quad M_n = G_{n_1} \times \cdots \times G_{n_s}, \quad M_{n+1} = G_{m_1} \times \cdots \times G_{m_t} \tag{3.1.1}$$

and

$$\pi = \pi_n \boxtimes \pi_{n+1}, \quad \pi_n = \pi_{n,1} \boxtimes \cdots \boxtimes \pi_{n,s}, \quad \pi_{n+1} = \pi_{n+1,1} \boxtimes \cdots \boxtimes \pi_{n+1,t}. \tag{3.1.2}$$

We say χ is *Rankin-Selberg regular*, if for any $1 \leq i \leq s, 1 \leq j \leq t$, we have $\pi_{n,i} \neq \pi^\vee_{n+1,j}$. We write $\mathfrak{X}_{\mathrm{RS}} \subset \mathfrak{X}([G])$ for the set of all the Rankin-Selberg regular cuspidal datum. We write $\mathcal{T}_{\mathrm{RS}}([G])$ (resp. $\mathcal{S}_{\mathrm{RS}}([G])$) for $\mathcal{T}_{\mathfrak{X}_{\mathrm{RS}}}([G])$ (resp. $\mathcal{S}_{\mathfrak{X}_{\mathrm{RS}}}([G])$).

3.1.3. *Zeta integral.* Recall that N_n is the unipotent radical of the Borel subgroup of G_n and we write $N = N_n \times N_{n+1}$. We define a character ψ'_N of $[N]$ by

$$\psi'_N(u, u') = \psi\left(-\sum_{i=1}^{n-1} u_{i,i+1} + \sum_{j=1}^{n} u'_{j,j+1}\right).$$

We write N_H for $N \cap H$. For $f \in \mathcal{T}([G])$, we associate

$$W'_f(g) = \int_{[N]} f(ug)\psi'_N(u)^{-1}\mathrm{d}u.$$

For $f \in \mathcal{T}([G])$ and $s \in \mathbb{C}$, we define the *zeta-integral* by

$$Z^{\mathrm{RS}}(s, f) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W'_f(h)|\det h|^s \mathrm{d}h,$$

provided by the integral is absolutely convergent. For any $f \in \mathcal{T}([G])$, the integral is convergent and holomorphic when $\mathrm{Re}(s) \gg 0$, see [BCZ22, Lemma 7.1.1.1].

3.1.4. *Main results.* The following theorem summarizes the main result of [BCZ22, §7].

Theorem 3.1.1. *([BCZ22, Theorem 7.1.3.1]) Let χ be a Rankin-Selberg regular cuspidal datum. Then*

(1) The linear functional $\mathcal{P}_{\mathrm{RS}}$ on $\mathcal{S}_\chi([G])$ extends (uniquely) by continuity to a linear functional $\mathcal{P}^*_{\mathrm{RS}}$ on $\mathcal{T}_\chi([G])$.

(2) For $f \in \mathcal{T}_\chi([G])$, the zeta integral $Z(\cdot, f)$ extends to an entire function of s. And we have

$$\mathcal{P}^*_{\mathrm{RS}}(f) = Z^{\mathrm{RS}}(0, f).$$

(3) For any $s \in \mathbb{C}$, the functional $Z^{\mathrm{RS}}(s, \cdot)$ on $\mathcal{T}_\chi([G])$ is continuous.

We provide a mild extension of Theorem 3.1.1 to $\mathcal{T}_{\mathrm{RS}}([G])$.

Proposition 3.1.2. *We have the following statements:*

(1) The linear functional $\mathcal{P}_{\mathrm{RS}}$ on $\mathcal{S}_{\mathrm{RS}}([G])$ extends (uniquely) by continuity to a linear functional $\mathcal{P}^*_{\mathrm{RS}}$ on $\mathcal{T}_{\mathrm{RS}}([G])$.

(2) For $f \in \mathcal{T}_{\mathrm{RS}}([G])$, the zeta integral $Z(\cdot, f)$ extends to an entire function of s. And we have

$$\mathcal{P}^*_{\mathrm{RS}}(f) = Z^{\mathrm{RS}}(0, f).$$

(3) For any $s \in \mathbb{C}$, the functional $Z^{\mathrm{RS}}(s, \cdot)$ on $\mathcal{T}_{\mathrm{RS}}([G])$ is continuous.

Proof. The proof follows the same line of [BCZ22, p. 300], we sketch the proof. For $f \in \mathcal{S}([G])$, we put

$$Z_n(s, f) = \int_{[H]} f(h)|\det h|^s \mathrm{d}h.$$

It is an entire function in s with the functional equation $Z(s, f) = Z(-s, \widetilde{f})$, where $\widetilde{f}(g) = f({}^t g^{-1})$. In order to apply Proposition 2.7.3, hence prove the proposition, it suffices to prove that

$$Z_n(s, f) = Z^{\mathrm{RS}}(s, f) \text{ for any } f \in \mathcal{S}_{\mathrm{RS}}([G]). \tag{3.1.3}$$

For $\chi \in \mathfrak{X}_{\mathrm{RS}}$, let f_χ be the projection of f in $\mathcal{S}_\chi([G])$. Then by Theorem 2.5.1, the sum $\sum_{\chi \in \mathfrak{X}_{\mathrm{RS}}} f_\chi$ is absolutely summable in $\mathcal{S}([G])$. By the main result of [BCZ22, §7], $Z^{\mathrm{RS}}(s, f_\chi) = Z_n(s, f_\chi)$ for any $\chi \in \mathfrak{X}_{\mathrm{RS}}$. By [BCZ22, Lemma 7.1.1.1], for any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \gg 0$, we have

$$\sum_{\chi \in \mathfrak{X}_{\mathrm{RS}}} Z^{\mathrm{RS}}(s, f_\chi) = Z^{\mathrm{RS}}(s, f).$$

where the RHS is absolutely summable. It is clear that $Z_n(s, f)$ depends continuously on f for any $s \in \mathbb{C}$, therefore

$$\sum_{\chi \in \mathfrak{X}_{\mathrm{RS}}} Z_n(s, f_\chi) = Z_n(s, f),$$

therefore (3.1.3) is proved. $\square$

We endow the topological dual $\mathcal{T}'_{\mathrm{RS}}([G])$ of $\mathcal{T}_{\mathrm{RS}}([G])$ with the weak topology. Since the natural map $\mathcal{T}_{\mathrm{RS}}([G]) \to (\mathcal{T}'_{\mathrm{RS}}([G]))'$ is a bijection, we obtain:

(3.1.4) The map $Z^{\mathrm{RS}}(\cdot,\cdot) : \mathbb{C} \to \mathcal{T}'_{\mathrm{RS}}([G]), s \mapsto (f \mapsto Z^{\mathrm{RS}}(s,f))$ is holomorphic.

3.2. **Rankin-Selberg period on $\mathrm{GL}_n \times \mathrm{GL}_n$.** In §3.2, we discuss the canonical extension of equal rank Rankin-Selberg based on [BLX24, §10.3]. The discussion is parallel to §3.1.

3.2.1. Let $G = \mathrm{GL}_n \times \mathrm{GL}_n$ and let $H = \mathrm{GL}_n$, regarded as the diagonal subgroup of G. For $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, the (equal rank) *Rankin-Selberg period* of f and Φ is defined by the absolute convergent integral

$$\mathcal{P}_{\mathrm{RS}}(f,\Phi) := \int_{[H]} f(h)\Theta(h,\Phi)\mathrm{d}h.$$

3.2.2. *Rankin-Selberg regular cuspidal datum.* Let $\chi \in \mathfrak{X}(G)$ be a cuspidal datum of G. Assume that χ is represented by (P,π), and we write

$$M_P = M_{P_1} \times M_{P_2}, \quad M_{P_1} = G_{n_1} \times \cdots \times G_{n_s}, \quad M_{P_2} = G_{m_1} \times \cdots \times G_{m_t} \tag{3.2.1}$$

and

$$\pi = \pi_1 \boxtimes \pi_2, \quad \pi_1 = \pi_{1,1} \boxtimes \cdots \boxtimes \pi_{1,s}, \quad \pi_2 = \pi_{2,1} \boxtimes \cdots \boxtimes \pi_{2,t}. \tag{3.2.2}$$

We say χ is *Rankin-Selberg regular*, if for any $1 \le i \le s, 1 \le j \le t$, we have $\pi_{1,i} \neq \pi_{2,j}^\vee$. We write $\mathfrak{X}_{\mathrm{RS}} \subset \mathfrak{X}([G])$ for the set of all the Rankin-Selberg regular cuspidal datum. We write $\mathcal{T}_{\mathrm{RS}}([G])$ (resp. $\mathcal{S}_{\mathrm{RS}}([G])$) for $\mathcal{T}_{\mathfrak{X}_{\mathrm{RS}}}([G])$ (resp. $\mathcal{S}_{\mathfrak{X}_{\mathrm{RS}}}([G])$).

3.2.3. *Zeta integral.* We define a character ψ'_N of $[N]$ by

$$\psi'_N(u,u') = \psi\left(-\sum_{i=1}^{n-1} u_{i,i+1} + \sum_{j=1}^{n-1} u'_{j,j+1}\right).$$

We write N_H for $N \cap H$. For $f \in \mathcal{T}([G])$, we associate

$$W'_f(g) = \int_{[N]} f(ug)\psi'_N(u)^{-1}\mathrm{d}u.$$

For $f \in \mathcal{T}([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_n)$ and $s \in \mathbb{C}$, we define the *zeta-integral* by

$$Z^{\mathrm{RS}}(s,f,\Phi) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W'_f(h)\Phi(e_n h)|\det h|^{s+\frac{1}{2}}\mathrm{d}h,$$

provided by the integral is absolutely convergent. For any $f \in \mathcal{T}([G])$, the integral is convergent when $\mathrm{Re}(s) \gg 0$ and holomorphic in s, see [BLX24, Lemma 10.2].

3.2.4. *Main results.*

Theorem 3.2.1. *([BLX24, Theorem 10.4, Lemma 10.5]) Let χ be a Rankin-Selberg regular cuspidal datum and $\Phi \in \mathcal{S}(\mathbb{A}_n)$. Then*

(1) The linear functional $\mathcal{P}_{\mathrm{RS}}(\cdot, \Phi)$ on $\mathcal{S}_\chi([G])$ extends (uniquely) by continuity to a linear functional $\mathcal{P}^*_{\mathrm{RS}}(\cdot, \Phi)$ on $\mathcal{T}_\chi([G])$.

(2) For $f \in \mathcal{T}_\chi([G])$, the zeta integral $Z(\cdot, f, \Phi)$ extends to an entire function of s. And we have

$$\mathcal{P}^*_{\mathrm{RS}}(f) = Z(0, f, \Phi).$$

(3) For any $s \in \mathbb{C}$, the bilinear form $Z(s, \cdot, \cdot)$ on $\mathcal{T}_\chi([G]) \times \mathcal{S}(\mathbb{A}_n)$ is continuous in the sense that there exists continuous semi-norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathcal{T}_\chi([G])$ and $\mathcal{S}(\mathbb{A}_n)$ respectively, such that

$$Z(s, f, \Phi) \ll \|f\| \|\Phi\|'.$$

Remark 3.2.2. In *loc.cit*, the theorem is stated for (G, H)-regular cuspidal datum, but the proof indeed works for general Rankin-Selberg regular cuspidal data. See also the proof of Lemma 4.5.1.

The following proposition is an analog of Proposition 3.1.2 and we omit the proof.

Proposition 3.2.3. *We have the following statements:*

(1) The linear functional $\mathcal{P}_{\mathrm{RS}}$ on $\mathcal{S}_{\mathrm{RS}}([G])$ extends (uniquely) by continuity to a linear functional $\mathcal{P}^*_{\mathrm{RS}}$ on $\mathcal{T}_{\mathrm{RS}}([G])$.

(2) For $f \in \mathcal{T}_{\mathrm{RS}}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, the zeta integral $Z(\cdot, f, \Phi)$ extends to an entire function of s.

(3) For any $s \in \mathbb{C}$, the bilinear map $Z(s, \cdot, \cdot)$ on $\mathcal{T}_{\mathrm{RS}}([G]) \times \mathcal{S}(\mathbb{A}_n)$ is continuous.

By the same argument of (3.1.4), we obtain that for any $\Phi \in \mathcal{S}(\mathbb{A}_n)$, the map $Z^{\mathrm{RS}}(\cdot, \Phi, \cdot) : \mathbb{C} \to \mathcal{T}'_{\mathrm{RS}}([G]), s \mapsto (f \mapsto Z^{\mathrm{RS}}(s, \Phi, f))$ is holomorphic.

Therefore, by (2.7.1), we see that

(3.2.3) The map $Z^{\mathrm{RS}}(\cdot, \cdot, \cdot) : \mathbb{C} \to \mathrm{Bil}_s(\mathcal{T}_{\mathrm{RS}}([G]), \mathcal{S}(\mathbb{A}_n); \mathbb{C}), s \mapsto ((f, \Phi) \mapsto Z^{\mathrm{RS}}(s, f, \Phi))$ is holomorphic.

3.2.5. *A twisted version.* Let $w_\ell = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in G_n$ be the longest Weyl group element. For $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, we define

$$\widetilde{\mathcal{P}}_{\mathrm{RS}}(f, \Phi) := \int_{[G_n]} f(w_\ell {}^t g^{-1} w_\ell, g) \Theta(g, \Phi) \mathrm{d}g.$$

For $f \in \mathcal{T}([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_n)$ and $s \in \mathbb{C}$, we put the twisted Zeta integral

$$\widetilde{Z}^{\mathrm{RS}}(s, f, \Phi) = \int_{N_n(\mathbb{A}) \backslash G_n(\mathbb{A})} W_f(w_\ell {}^t g^{-1} w_\ell, g) \Phi(e_n g) |\det g|^{s+\frac{1}{2}} \mathrm{d}g,$$

provided by the integral is absolutely convergent.

Let $\chi \in \mathfrak{X}(G)$. Assume that χ is represented by (M, π) where M and π are as in (3.2.1) and (3.2.2). We say that χ is *twisted Rankin-Selberg regular*, if for any $1 \leq i \leq s, 1 \leq j \leq t$, we have $\pi_{1,i} \neq \pi_{2,j}$. Let $\widetilde{\mathfrak{X}}_{\mathrm{RS}} \subset \mathfrak{X}(G)$ denote the set of twisted Rankin-Selberg regular cuspidal datum. We write $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$ for $\mathcal{T}_{\widetilde{\mathfrak{X}}_{\mathrm{RS}}}([G])$.

Corollary 3.2.4. *We have the following statements:*

(1) *For $f \in \mathcal{T}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, there exists $C > 0$ such that the integral defining $\widetilde{Z}^{\mathrm{RS}}(s, f, \Phi)$ is convergent for* $\mathrm{Re}(s) > C$ *and defines a holomorphic function on $\mathcal{H}_{>C}$.*

(2) *For any $\Phi \in \mathcal{S}(\mathbb{A}_n)$, the linear functional $\widetilde{\mathcal{P}}_{\mathrm{RS}}(\cdot, \Phi)$ on $\mathcal{S}_{\widetilde{\mathrm{RS}}}([G])$ extends (uniquely) by continuity to a continuous linear functional $\widetilde{\mathcal{P}}_{\mathrm{RS}}(\cdot, \Phi)$ on $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$.*

(3) *For any $f \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, the zeta integral $\widetilde{Z}^{\mathrm{RS}}(\cdot, f, \Phi)$ extends to an entire function. And we have*

$$\widetilde{\mathcal{P}}_{\mathrm{RS}}(f, \Phi) = \widetilde{Z}^{\mathrm{RS}}(0, f, \Phi)$$

(4) *For any $s \in \mathbb{C}$, the bilinear map $\widetilde{Z}^{\mathrm{RS}}(s, \cdot, \cdot)$ on $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G]) \times \mathcal{S}(\mathbb{A}_n)$ is continuous.*

Proof. For a function f on $[G]$, we put a new function f' by $f'(g_1, g_2) = f(w_\ell {}^t g_1^{-1} w_\ell, g_2)$. Then $f \in \mathcal{S}([G])$ (resp. $\mathcal{T}([G])$) if and only if $f' \in \mathcal{S}([G])$ (resp. $f' \in \mathcal{T}([G])$). Moreover, $f \mapsto f'$ induces an isomorphism of $\mathcal{S}([G])$ (resp. $\mathcal{T}([G])$) to itself.

Note that for $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, we have $\mathcal{P}_{\mathrm{RS}}(f, \Phi) = \widetilde{\mathcal{P}}_{\mathrm{RS}}(f', \Phi)$. For $f \in \mathcal{T}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$, we have $Z^{\mathrm{RS}}(s, f, \Phi) = \widetilde{Z}^{\mathrm{RS}}(s, f', \Phi)$. The corollary then easily follows from Proposition 3.1.2. ☐

By (3.2.3), we see that

(3.2.4) The map $\widetilde{Z}^{\mathrm{RS}}(\cdot, \cdot, \cdot) : \mathbb{C} \to \mathrm{Bil}_s(\mathcal{T}_{\widetilde{\mathrm{RS}}}([G]), \mathcal{S}(\mathbb{A}_n); \mathbb{C}), s \mapsto ((f, \Phi) \mapsto \widetilde{Z}^{\mathrm{RS}}(s, f, \Phi))$ is holomorphic.

3.2.6. *Euler decomposition.* Let $\mathtt{S}$ be a finite set of places of F, let $\sigma = \sigma_n \boxtimes \sigma_n'$ be a generic irreducible representation of $G(F_{\mathtt{S}})$. For $W \in \mathcal{W}(\sigma, \psi_{N,\mathtt{S}})$ and $\Phi \in \mathcal{S}(F_{\mathtt{S}})$, we define local (twisted) Rankin-Selberg integral [JPS83] as

$$\widetilde{Z}_{\mathtt{S}}^{\mathrm{RS}}(s, W, \Phi) := \int_{N_n(F_{\mathtt{S}}) \backslash G_n(F_{\mathtt{S}})} W(w_\ell h^{-1} w_\ell, h) \Phi(e_n h) |\det h|^{s+\frac{1}{2}} \mathrm{d}h.$$

The integral defining $\widetilde{Z}_{\mathtt{S}}^{\mathrm{RS}}(s, W, \Phi)$ is convergent when $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $\mathbb{C}$. Moreover, by [JPS83] and [Jac09], for any $W \in \mathcal{W}(\sigma, \psi_{N,\mathtt{S}})$ and $\Phi \in \mathcal{S}(F_{\mathtt{S}})$, the quotient

$$\frac{\widetilde{Z}_{\mathtt{S}}^{\mathrm{RS}}(s, W, \Phi)}{L_{\mathtt{S}}(s + \frac{1}{2}, \sigma_n^\vee \times \sigma_n')}$$

is entire.

Let $P \subset G$ be a standard parabolic subgroup and let π be a cuspidal automorphic representation of M_P. Assume that (M_P, π) gives a twisted Rankin-Selberg regular cuspidal data χ. Let $\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi = \Pi_n \boxtimes \Pi'_n$. Let $\varphi \in \Pi$ and $\Phi \in \mathcal{S}(\mathbb{A}_n)$.

Let $\mathtt{S}$ be a sufficiently large set of places of F, that we assume to contain Archimedean places as well as the places where Π, ψ, φ or Φ is ramified. We then have a decomposition $W_{E(\varphi)} = W_{E(\varphi),\mathtt{S}} W_{E(\varphi)}^{\mathtt{S}}$ such that $W_{E(\varphi)}^{\mathtt{S}}(1) = 1$ and is fixed by $K^{\mathtt{S}}$. We also write Φ as $\Phi = \Phi_{\mathtt{S}} \Phi^{\mathtt{S}}$, where $\Phi^{\mathtt{S}}$ is the characteristic function of $\mathcal{O}_F^{\mathtt{S}}$ and $\Phi_{\mathtt{S}} \in \mathcal{S}(F_{\mathtt{S}})$.

By the unramified computation for the Rankin-Selberg integral, we have

$$\widetilde{Z}^{\mathrm{RS}}(s, E(\varphi), \Phi) = (\Delta_{G_n}^{\mathtt{S},*})^{-1} \widetilde{Z}_{\mathtt{S}}^{\mathrm{RS}}(s, W_{E(\varphi),\mathtt{S}}, \Phi_{\mathtt{S}}) L^{\mathtt{S}}(s + \frac{1}{2}, \Pi_n^{\vee} \times \Pi'_n). \tag{3.2.5}$$

4. Canonical extensions of Rankin-Selberg periods – higher corank

4.1. Statements of main results.

4.1.1. *Notations.* In §4, $n \geq 0, m \geq 2$ be integers. Let $G = G_n \times G_{n+m}$. Let $H = G_n$ be the subgroup of G consisting of matrices of the form $(g, \mathrm{diag}(g, 1_m))$.

For integers $0 \leq r \leq k$, let $N_{r,k}$ be the unipotent radical of the standard parabolic subgroup of G_k with Levi $G_r \times G_1^{k-r}$. Note that $N_{0,k} = N_{1,k}$ is the upper triangular unipotent subgroup of G_k and $N_{k,k} = \{1\}$.

For $0 \leq r \leq n$, we then put

$$N_r^G := N_{r,n} \times N_{r,n+m}, \quad N_r^H := N_r^G \cap H \cong N_{r,n}.$$

In particular, $N := N_0^G$ is a maximal unipotent subgroup of G and $N_H := N \cap H = N_0^H$ is a maximal unipotent subgroup of H.

We also put

$$N_{n+1}^G := 1 \times N_{n+1,n+m}.$$

We define a character ψ_N of $[N]$ by

$$\psi'_N(u, u') = \psi\left(-\sum_{i=1}^{n-1} u_{i,i+1} + \sum_{j=1}^{n+m-1} u'_{j,j+1}\right), \quad u \in [N_n], u' \in [N_{n+m}].$$

For $1 \leq r \leq n+1$, ψ'_N restricts to a character on the subgroup N_r^G, we denote it by ψ'_r.

4.1.2. *Rankin-Selberg regular cuspidal datum.* Let $\chi \in \mathfrak{X}(G)$, assume that χ is represented by (M, π), where we write

$$M = M_n \times M_{n+m}, \quad M_n = G_{n_1} \times \cdots \times G_{n_s}, \quad M_{n+m} = G_{m_1} \times \cdots \times G_{m_t}, \tag{4.1.1}$$

and

$$\pi = \pi_n \boxtimes \pi_{n+m}, \quad \pi_n = \pi_{n,1} \boxtimes \cdots \boxtimes \pi_{n,s}, \quad \pi_{n+m} = \pi_{n+m,1} \boxtimes \cdots \boxtimes \pi_{n+m,t}. \tag{4.1.2}$$

we say that χ is *Rankin-Selberg regular*, for any $1 \leq i \leq s$ and $1 \leq j \leq k$ we have $\pi_{n,i} \neq \pi_{n+m,j}^\vee$.

Let $\mathfrak{X}_{\mathrm{RS}}$ denote the set of Rankin-Selberg regular cuspidal datum. We write $\mathcal{S}_{\mathrm{RS}}([G])$ (resp. $\mathcal{T}_{\mathrm{RS}}([G])$) for $\mathcal{S}_{\mathfrak{X}_{\mathrm{RS}}}([G])$ (resp. $\mathcal{T}_{\mathrm{RS}}([G])$).

4.1.3. *Zeta integrals.* For $f \in \mathcal{T}([G])$, let

$$W'_f(g) = \int_{[N]} f(ug)\psi'_N(u)^{-1}\mathrm{d}u$$

be its Whittaker model. For $s \in \mathbb{C}$, we put

$$Z^{\mathrm{RS}}(s,f) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W'_f(h)|\det h|^s \mathrm{d}h$$

provided by the integral is absolutely convergent.

Lemma 4.1.1. *For any $N > 0$, then there exist $c_N > 0$ such that*

- *(1) For every $f \in \mathcal{T}_N([G])$, the integral defining $Z^{\mathrm{RS}}(s,f)$ is absolutely convergent for* $\mathrm{Re}(s) > c_N$, *and $Z(\cdot,f)$ is holomorphic and bounded in vertical strips on $\mathcal{H}_{>c_N}$.*
- *(2) For every $s \in \mathcal{H}_{>c_N}$, the functional $f \mapsto Z^{\mathrm{RS}}(s,f)$ is continuous.*

The proof the Lemma 4.1.1 will be given in §4.4.4.

4.1.4.

Theorem 4.1.2. *We have the following assertions:*

- *(1) For any $f \in \mathcal{S}([G])$, the* Rankin-Selberg period

$$\mathcal{P}_{\mathrm{RS}}(f) := \int_{[H]} f_{N_{n+1}^G,\psi'_{n+1}}(h)\mathrm{d}h$$

  is absolutely convergent.
- *(2) The restriction of $\mathcal{P}_{\mathrm{RS}}$ to $\mathcal{S}_{\mathrm{RS}}([G])$ extends by continuity to a linear functional $\mathcal{P}^*_{\mathrm{RS}}$ on $\mathcal{T}_{\mathrm{RS}}([G])$.*
- *(3) For $f \in \mathcal{T}_{\mathrm{RS}}([G])$, the zeta integral $Z(\cdot,f)$ extends to an entire function of s. And we have*

$$\mathcal{P}^*_{\mathrm{RS}}(f) = Z^{\mathrm{RS}}(0,f)$$

- *(4) For any $s \in \mathbb{C}$, the functional $Z^{\mathrm{RS}}(s,\cdot)$ is continuous.*

The rest of §4 is devoted to the proof of Theorem 4.1.2.

4.2. **Proof of Theorem 4.1.2.**

4.2.1. *An unfolding identity.* For $f \in \mathcal{S}([G])$ and $s \in \mathbb{C}$, we put

$$Z_{n+1}(s, f) = \int_{[H]} f_{N^G_{n+1}, \psi'_{n+1}}(h) |\det h|^s \mathrm{d}h. \tag{4.2.1}$$

Lemma 4.2.1. *For any $f \in \mathcal{S}([G])$, the integral defining $Z_{n+1}(s, f)$ is absolutely convergent for any $s \in \mathbb{C}$, and $s \mapsto Z_{n+1}(s, f)$ is entire. Moreover, for any $s \in \mathbb{C}$, the map $f \mapsto Z_{n+1}(s, f)$ is continuous on $\mathcal{S}([G])$.*

The proof of the Lemma 4.2.1 will be given in §4.4.2.

Proposition 4.2.2. *Let χ be an Rankin-Selberg regular cuspidal data. Then for any $f \in \mathcal{S}_\chi([G])$, we have*

$$Z_{n+1}(s, f) = Z^{\mathrm{RS}}(s, f)$$

holds for $\mathrm{Re}(s)$ *sufficiently large.*

The proof of Proposition 4.2.2 will be given in §4.5.

Corollary 4.2.3. *Let $f \in \mathcal{S}_{\mathrm{RS}}([G])$. For* $\mathrm{Re}(s) \gg 0$, *we have*

$$Z_{n+1}(s, f) = Z^{\mathrm{RS}}(s, f)$$

holds for $\mathrm{Re}(s)$ *sufficiently large.*

Proof. By Theorem 2.5.1, we have a decomposition

$$f = \sum_{\chi \in \mathfrak{X}_{\mathrm{RS}}} f_\chi,$$

where $f_\chi \in \mathcal{S}_\chi([G])$ and the sum is absolutely summable in $\mathcal{S}([G])$. By Lemma 4.1.1 and Lemma 4.2.1, both $Z_{n+1}(s, \cdot)$ and $Z^{\mathrm{RS}}(s, \cdot)$ is continuous when $\mathrm{Re}(s)$ is large enough. The result follows. $\square$

4.2.2. *Another zeta integral.* For $f \in \mathcal{T}([G])$, we put

$$W''_f(g) := \int_{[N]} f(ug) \psi'_N(u) \mathrm{d}u, \quad g \in G_{n+m}(\mathbb{A}).$$

Then we define

$$Z'_1(s, f) = \int_{N_H(\mathbb{A}) \backslash H(\mathbb{A})} \int_{\mathrm{Mat}_{(m-1) \times n}(\mathbb{A})} W''_f \left(h, \begin{pmatrix} 1_n & & \\ x & 1_{m-1} & \\ & & 1 \end{pmatrix} h \right) |\det h|^s \mathrm{d}x \mathrm{d}h. \tag{4.2.2}$$

provided by the integral is absolutely convergent.

Lemma 4.2.4. *For any $N > 0$, there exists $c_N > 0$ such that*

(1) *For any $f \in \mathcal{T}_N([G])$, the double integral defining $Z'_1(s, f)$ is absolutely convergent for* $\mathrm{Re}(s) > c_N$, *and $Z(\cdot, f)$ is holomorphic and bounded in vertical strips on $\mathcal{H}_{>c_N}$.*
(2) *For every $s \in \mathcal{H}_{>c_N}$, the functional $f \mapsto Z'_1(s, f)$ is continuous.*

The proof of the Lemma 4.2.4 will be given in §4.4.5.

4.2.3. *Another unfolding identity.* Let $w_{\ell,m}$ denote the matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ of size m. Let $w_{n,m}$ denote the matrix $\begin{pmatrix} 1_n & \\ & w_{\ell,m} \end{pmatrix}$. For a function f on $[G]$, we put $\widetilde{f}(g) := f({}^t g^{-1})$.

Proposition 4.2.5. *Let χ be an Rankin-Selberg regular cuspidal data. Then for any $f \in \mathcal{S}_\chi([G])$, we have*

$$Z_{n+1}(s,f) = Z_1'(-s, R(w_{n,m})\widetilde{f}),$$

when $\mathrm{Re}(s)$ *is sufficiently large.*

The proof of Proposition 4.2.5 will be given in §4.5.

By the same argument of Corollary 4.2.3,

(4.2.3) $Z_{n+1}(s,f) = Z_1'(-s, R(w_{n,m})\widetilde{f})$, holds for any $f \in \mathcal{S}_{\mathrm{RS}}([G])$.

4.2.4. *Proof of Theorem 4.1.2.* Assertion (1) is a special case Lemma 4.2.1. Fix $N > 0$, we apply Proposition 2.7.3 to

$$W = L^2_{N,\mathrm{RS}}([G])^\infty, \quad S = \mathcal{S}_{\mathrm{RS}}([G]), \quad Z_+(s,f) = Z^{\mathrm{RS}}(s,f), \quad Z_-(s,f) = Z_1'(s, R(w_{n,m})\widetilde{f}).$$

The conditions of Proposition 2.7.3 are satisifed by Lemma 4.1.1, Lemma 4.2.4, Lemma 4.2.1, Corollary 4.2.3 and (4.2.3).

As a consequence, for any $f \in L^2_{N,\mathrm{RS}}([G])^\infty$, $Z^{\mathrm{RS}}(s,f)$ is entire and for any $s \in \mathbb{C}$, the map $f \mapsto Z^{\mathrm{RS}}(s,f)$. As N varies, Assertion (4) is proved.

For $f \in L^{2,\infty}_N([G])^\infty$, we put

$$\mathcal{P}^*_{\mathrm{RS}}(f) := Z(0,f),$$

by Corollary (4.2.3), $\mathcal{P}^*_{\mathrm{RS}}$ defines a continuous extension of $\mathcal{P}_{\mathrm{RS}}$ to $L^2_N([G])^\infty$. As N varies, assertions (2) and (3) are proved.

4.2.5. We endow the topological dual $\mathcal{T}'_{\mathrm{RS}}([G])$ with the weak topology, from Theorem 4.1.2, we see that

(4.2.4) The map $Z^{\mathrm{RS}}(\cdot,\cdot) : \mathbb{C} \to \mathcal{T}'_{\mathrm{RS}}([G]), s \mapsto (f \mapsto Z^{\mathrm{RS}}(s,f))$ is holomorphic.

4.3. **Exchange of root identity.** We prove an exchange of root identity in the style of [MS11, Appendix A], [IT13, §4]. The main result is Corollary 4.3.3.

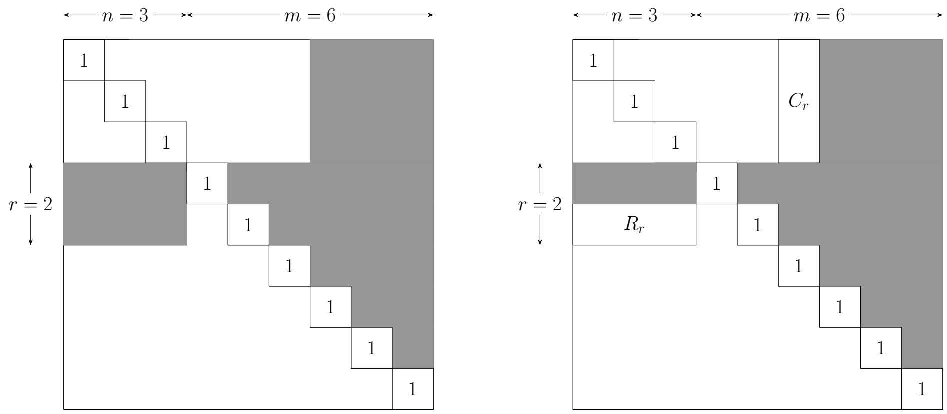

FIGURE 1. The unipotent subgroups $\mathcal{U}_r$, $\mathcal{U}'_r$, R_r and C_r

4.3.1. *Settings.* For $0 \leq r \leq m-1$, let $\mathcal{U}_r$ denote the unipotent subgroup of G_{n+m} of the shape in the left of figure 4.3.1.

It consists of matrices (u_{ij}) with 1 on the diagonal and $u_{ij} \neq 0$ only when $j > i > n$ or $1 \leq i \leq n, j \geq n+r+2$ or $1 \leq j \leq n$ and $n+1 \leq i \leq n+r$. Note that $\mathcal{U}_0 = N_{n+1,n+m}$

Let ψ_r denote the character $(u_{ij}) \mapsto \psi(u_{n+1,n+2} + \cdots + u_{n+m-1,n+m})$ on $\mathcal{U}_r(\mathbb{A})$.

For $r \geq 1$ and $x \in \mathbb{A}_n$, let $R_r(x)$ denote the matrix $\begin{pmatrix} 1_n & & & \\ & 1_{r-1} & & \\ x & & 1 & \\ & & & 1_{m-r} \end{pmatrix}$. We write R_r for the algebraic subgroup of G_{n+m} formed by $R_r(x)$.

4.3.2.

Lemma 4.3.1. *Let $1 \leq r \leq m-1$ and $1 \leq k \leq r$. Let $f \in \mathcal{T}([G_{n+m}])$. The integral*

$$\int_{\mathrm{Mat}_{k\times n}(\mathbb{A})} f_{\mathcal{U}_{r-k},\psi_{r-k}}\left(\begin{pmatrix} 1_n & & & \\ & 1_{r-k} & & \\ x & & 1_k & \\ & & & 1_{m-r} \end{pmatrix} g\right) \mathrm{d}x \tag{4.3.1}$$

is absolutely convergent for any $g \in G_{n+m}(\mathbb{A})$.

The proof of Lemma 4.3.1 will be given in §4.3.5.

4.3.3.

Lemma 4.3.2. *Let $f \in \mathcal{T}([G_{n+m}])$ and $1 \leq r \leq m-1$, then*

$$f_{\mathcal{U}_r,\psi_r}(g) = \int_{\mathbb{A}_n} f_{\mathcal{U}_{r-1},\psi_{r-1}}(R_r(x)g)\mathrm{d}x \tag{4.3.2}$$

holds for any $g \in G_{n+m}(\mathbb{A})$, where the integral on the right-hand side is convergent.

Proof. The convergence of the integral follows from Lemma 4.3.1. For $1 \leq r \leq m-1$, let $\mathcal{U}'_r := \mathcal{U}_r \cap \mathcal{U}_{r-1}$ denote the subgroup of $\mathcal{U}_r$, see the shaded region of right hand side of figure 4.3.1. Let ψ'_r denote the restriction of ψ_r (equivalently ψ_{r-1}) on $\mathcal{U}'_r$.

For $y \in \mathbb{A}^n$, let $C_r(y)$ denote the element $\begin{pmatrix} 1_n & & y & \\ & 1_{r-1} & & \\ & & 1 & \\ & & & 1_{m-r} \end{pmatrix}$ of $G_{n+m}(\mathbb{A})$. Let C_r denote the algebraic subgroup of G_{n+m} formed by $C_r(y)$.

The following statements can be checked directly:

(1) $\mathcal{U}_r = \mathcal{U}'_r \rtimes R_r$, $\mathcal{U}_{r-1} = \mathcal{U}'_r \rtimes C_r$, R_r normalizes $\mathcal{U}_{r-1}$ and C_r normalizes $\mathcal{U}_r$.

(2) By (1) above, we can write an element of $\mathcal{U}_{r-1}(\mathbb{A})$ by $u'C_r(y)$, where $u' \in \mathcal{U}'_r(\mathbb{A})$ and $y \in \mathbb{A}^n$. For $a \in F_n$, the map $u'C_r(y) \mapsto \psi_r(u')\psi(ay)$ defines a character of $\mathcal{U}_{r-1}(\mathbb{A})$ trivial on $\mathcal{U}_{r-1}(F)$. We denote this character by $\psi_{r-1,a}$. Note that $\psi_{r-1,0} = \psi_{r-1}$.

(3) The equality

$$\psi_{r-1}(R_r(-a)uR_r(a)) = \psi_{r-1,-a}(u) \tag{4.3.3}$$

holds for any $a \in F_n, u \in \mathcal{U}_{r-1}(\mathbb{A})$.

(4) As a consequence of (3) above, the equation

$$\psi'_r(R_r(-a)uR_r(a)) = \psi'_r(u) \tag{4.3.4}$$

holds for any $a \in F_n$ and $u \in \mathcal{U}'_r(\mathbb{A})$. Similarly, one can check

$$\psi'_r(C_r(-b)uC_r(b)) = \psi'_r(u) \tag{4.3.5}$$

holds for any $b \in F^n$ and $u \in \mathcal{U}'_r(\mathbb{A})$.

By (4.3.4) above, for any $g \in G_{n+1}(\mathbb{A})$, we have

$$f_{\mathcal{U}_r,\psi_r}(g) = \int_{F_n\backslash\mathbb{A}_n} f_{\mathcal{U}'_r,\psi'_r}(R_r(x)g)\mathrm{d}x. \tag{4.3.6}$$

By (4.3.5), for any $g \in G_{n+m}(\mathbb{A})$, the map $y \mapsto f_{\mathcal{U}'_r,\psi'_r}(C_r(y)g)$ defines a function on $F^n\backslash\mathbb{A}^n$. Therefore, by Fourier expansion, we can write

$$f_{\mathcal{U}'_r,\psi'_r}(g) = \sum_{a\in F_n} \int_{F^n\backslash\mathbb{A}^n} f_{\mathcal{U}'_r,\psi'_r}(C_r(y)g)\psi^{-1}(ay)\mathrm{d}y = \sum_{a\in F_n} f_{\mathcal{U}_{r-1},\psi_{r-1,-a}}(g) \tag{4.3.7}$$

By (4.3.3), we have

$$f_{\mathcal{U}_{r-1},\psi_{r-1,-a}}(g) = f_{\mathcal{U}_{r-1},\psi_{r-1}}(R_r(a)g). \tag{4.3.8}$$

Combining (4.3.6), (4.3.7) and (4.3.8) and Lemma 4.3.1, (4.3.2) is proved. □

4.3.4.

Corollary 4.3.3. *For any $f \in \mathcal{T}([G_{n+m}])$, we have*

$$f_{\mathcal{U}_{m-1},\psi_{m-1}}(g) = \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} f_{\mathcal{U}_0,\psi_0}\left(\begin{pmatrix} 1_n & & \\ x & 1_{m-1} & \\ & & 1 \end{pmatrix} g\right) \mathrm{d}x,$$

where the integral of the right-hand side is absolutely convergent.

Proof. The convergence of the right-hand side follows from Lemma 4.3.1.

The equality follows from successively using Lemma 4.3.2. The convergence of each step also follows from Lemma 4.3.1. □

4.3.5. *Convergence.*

Proof of Lemma 4.3.1. We temporarily denote by P the standard parabolic subgroup of G_{n+m} whose Levi factor is $G_1^{r-k} \times G_n \times G_1^{m-r+k}$. Let ψ_{N_P} denote the character

$$(u_{ij}) \mapsto \psi(u_{12} + \cdots + u_{r-k-1,r-k} + u_{r-k,n+r-k+1} + u_{n+r-k+1,n+r-k+2} + \cdots + u_{n+m-1,n+m})$$

on $N_P(\mathbb{A})$. When $r-k=0$, this is understood as $(u_{ij}) \mapsto \psi(u_{n,n+1} + \cdots + u_{n+m-1,n+m})$. When $r-k=1$, this is understood as $(u_{ij}) \mapsto \psi(u_{1,n+2} + u_{n+2,n+3} + \cdots + u_{n+m-1,n+m})$.

Let $w \in G_{n+m}(F)$ be the permutation matrix associated to the permutation sending $1, 2, \cdots, n+m$ to $n+1, \cdots, n+r-k, 1, \cdots, n, n+r-k+1, \cdots, n+m$ respectively. Then the right hand side of (4.3.1) can be written as

$$\int_{\mathrm{Mat}_{k\times n}(\mathbb{A})} f_{N_P,\psi_{N_P}}\left(\begin{pmatrix} 1_{r-k} & & & \\ & 1_n & & \\ & x & 1_k & \\ & & & 1_{m-r} \end{pmatrix} gw\right) \mathrm{d}x. \tag{4.3.9}$$

Therefore we are reduced to show the convergence of (4.3.9). Let Q be the parabolic subgroup of G_{n+k} whose Levi factor is $G_n \times (G_1)^k$. Assume that $\begin{pmatrix} 1_n & \\ x & 1_k \end{pmatrix}$ is written as (2.2.2), then we have the Iwasawa decomposition

$$\begin{pmatrix} 1_{r-k} & & & \\ & 1_n & & \\ & x & 1_k & \\ & & & 1_{m-r} \end{pmatrix} = u'(x) \begin{pmatrix} 1_{r-k} & & & \\ & g(x) & & \\ & & t(x) & \\ & & & 1_{m-r} \end{pmatrix} k'(x)$$

for some $(u'(x), k'(x)) \in N_P(\mathbb{A}) \times K_{n+m}$. Write $t(x)$ as $\mathrm{diag}(t_1(x), \cdots, t_k(x))$, assume that $f \in \mathcal{T}_N([G_{n+m}])$, by Lemma 2.3.5, we see that the integral (4.3.9) is essentially bounded by

$$\int_{\mathrm{Mat}_{k\times n}(\mathbb{A})} \prod_{i=1}^{k-1} \|t_i(x)t_{i+1}(x)^{-1}\|_{\mathbb{A}}^{-N_1} \|t_k(x)\|_{\mathbb{A}}^{-N_1} \prod_{i=1}^{k} \|t_i(x)\|_{G_1}^{N} \|g\|_{G_n}^{N} \mathrm{d}x \tag{4.3.10}$$

for any $N_1 > 0$. Note that for any $N_2 > 0$, there exists $N_1 > 0$, such that

$$\prod_{i=1}^{k-1} \|t_i(x)t_{i+1}(x)^{-1}\|_{\mathbb{A}}^{-N_1} \|t_k(x)\|_{\mathbb{A}}^{-N_1} \ll \prod_{i=1}^{k} \|t_i(x) \cdots t_n(x)\|^{-N_2}. \tag{4.3.11}$$

Combining (4.3.11), (2.2.3) and (2.2.4), we see that the integral (4.3.10) is bounded by

$$\int_{\mathrm{Mat}_{k\times n}(\mathbb{A})} \|x\|_{\mathrm{Mat}_{k\times n}(\mathbb{A})}^{-N_2} \mathrm{d}x$$

for any $N_2 \gg 0$. The convergence hence follows. $\square$

4.4. **Convergence of zeta integrals.**

4.4.1. *More zeta integrals.* The goal of §4.4 is to prove convergence of various zeta integrals.

For later use in §4.5, we introduce more zeta integrals. Let $f \in \mathcal{S}([G])$. For $1 \le r \le n+1$, we define

$$Z_r(s,f) = \int_{\mathcal{P}_r(F)N_r^H(\mathbb{A})\backslash H(\mathbb{A})} f_{N_r^G,\psi_r'}(h)|\det h|^s \mathrm{d}h.$$

Note that when $r = n+1$, this coincides with the definition in (4.2.1), and $Z_1(s,f) = Z^{\mathrm{RS}}(s,f)$

For $1 \le r \le n$, we also introduce

$$Z_r'(s,f) = \int_{\mathcal{P}_r(F)N_{r,n}(\mathbb{A})\backslash G_n(\mathbb{A})} \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} f_{N_r^G,\psi_r'^{,-1}} \left(h, \begin{pmatrix} 1_n & & \\ x & 1_{m-1} & \\ & & 1 \end{pmatrix} h \right) |\det h|^s \mathrm{d}h.$$

Note that when $r = 1$, this coincides with the definition in (4.2.2). And we put

$$Z_{n+1}'(s,f) = \int_{[G_n]} \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} f_{N_{n+1}^G,\psi_{n+1}'^{,-1}} \left(h, \begin{pmatrix} 1_n & & \\ x & 1_{m-1} & \\ & & 1 \end{pmatrix} h \right) |\det h|^s \mathrm{d}h.$$

Lemma 4.4.1. *For $f \in \mathcal{S}([G])$ and $1 \le r \le n+1$, the integral defining $Z_r(s,f)$ and $Z_r'(s,f)$ are absolutely convergent when* $\mathrm{Re}(s)$ *is sufficiently large.*

4.4.2. *Proof of Lemma 4.2.1.* By Lemma 2.3.2, for any $N, M > 0$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}([G])$, such that the integral defining $Z_{n+1}(s,f)$ is bounded by

$$\|f\| \cdot \int_{[H]} \|h\|_H^{-N} \|h\|_H^{-M} \delta_{P_{n+1,n+m}}(h)^{-cM} |\det h|^{\mathrm{Re}(s)} \mathrm{d}h$$

for some constants $c > 0$ and any $N, M > 0$. The result follows.

4.4.3. *Proof of Lemma 4.4.1.* Let Q_r denote the parabolic subgroup of H with Levi component $G_r \times G_1^{n-r}$. Convergence of $Z_{n+1}(s,f)$ is covered in Lemma 4.2.1. For $1 \le r \le n$, let P_r denote the parabolic subgroup of G with Levi component $(G_r \times G_1^{n-r}) \times (G_r \times G_1^{m+n-r})$. Using Iwasawa decomposition $H(\mathbb{A}) = Q_r(\mathbb{A})K_n$. The integral defining $Z_r(s,f)$ is bounded by

$$\int_{\mathcal{P}_r(F)\backslash G_r(\mathbb{A})} \int_{[G_1]^{n-r}} \int_{K_n} \left| R(k) f_{N_r^G,\psi_r'} \begin{pmatrix} h & \\ & t \end{pmatrix} \right| |\det h|^s |\det t|^s \delta_{Q_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} \mathrm{d}h \mathrm{d}t \mathrm{d}k. \quad (4.4.1)$$

By Lemma 2.3.5, there exists $c > 0$ such that for any $N > 0$ and $N_1 > 0$, we have

$$\left| R(k) f_{N_r^G,\psi_r'} \begin{pmatrix} h & \\ & t \end{pmatrix} \right| \ll$$

$$\|t_1^{-1} e_r h_r\|_{\mathbb{A}_r}^{-2N_1} \|t_1 t_2^{-1}\|_{\mathbb{A}}^{-2N_1} \cdots \|t_{n-r-1} t_{n-r}^{-1}\|_{\mathbb{A}}^{-2N_1} \|t_{n-r}\|_{\mathbb{A}}^{-N_1} \delta_{P_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-cN} \|h\|_{G_r}^{-N} \|t\|_{G_1^r}^{-N},$$

where $t = \mathrm{diag}(t_1, \cdots, t_{n-r})$. Since for any $N_2 > 0$ there exists $N_1 > 0$ such that

$$\|t_1^{-1} e_r h_r\|_{\mathbb{A}_r}^{-2N_1} \|t_1 t_2^{-1}\|_{\mathbb{A}}^{-2N_1} \cdots \|t_{n-r-1} t_{n-r}^{-1}\|_{\mathbb{A}}^{-2N_1} \|t_{n-r}\|_{\mathbb{A}}^{-N_1} \ll \|e_r h\|_{\mathbb{A}}^{-N_2} \|t_1\|_{\mathbb{A}}^{-N_2} \cdots \|t_{n-r}\|_{\mathbb{A}}^{-N_2}.$$

We then see that the integral (4.4.1) is essentially bounded by

$$\int_{\mathcal{P}_r(F)\backslash G_r(\mathbb{A})} \int_{[G_1]^{n-r}} \|e_r h\|_{\mathbb{A}}^{-N_2} \prod_{i=1}^{n-r} \|t_i\|_{\mathbb{A}}^{-N_2} |\det h|^{s-\alpha(N)} \prod_{i=1}^{t-r} |t_i|^{s-\alpha_i(N)} \|h\|_{G_n}^{-N} \mathrm{d}h \mathrm{d}t, \quad (4.4.2)$$

for some $c > 0$ and any $N > 0$ and $N_2 > 0$, where

$$\alpha(N) = 2c(n+m-r)N + n - r, \quad \alpha_i(N) = c(2n+m-4r+2-4i)N + (n-2r+1-2i). \quad (4.4.3)$$

We have $\alpha(N) > \alpha_1(N) > \cdots > \alpha_{n-r}(N)$. Let $C = 2$ in Corollary 2.3.4, together with Corollary 2.2.5, we see that there exists $N_0 > 0$ such that for any $C_1 > 0$ and $N > N_0$, the integral is convergent for

$$-2 + \alpha(N) < \mathrm{Re}(s) < 2 + \alpha(N) \text{ and } 1 + \alpha_1(N) < \mathrm{Re}(s) < C_1 + \alpha_{n-r}(N).$$

As N and C_1 vary, we see that the integral is convergent when $\mathrm{Re}(s) \gg 1$. This shows Lemma 4.4.1 for $Z_r(s,f)$.

To show the convergence of $Z_r'(s,f)$. We prove the case when $1 \le r \le n$, the case when $r = n+1$ follows from a similar (and easier) argument. For simplicity, for $x \in \mathrm{Mat}_{(m-1)\times n}$ we write $A(x)$ for the matrix $\begin{pmatrix} 1_n & & \\ x & 1_{m-1} & \\ & & 1 \end{pmatrix}$. The absolute convergence of $Z_r'(s,f)$ is equivalent to the convergence of

$$\int_{\mathcal{P}_r(F)N_{r,n}(\mathbb{A})\backslash G_n(\mathbb{A})} \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} \left| f_{N_r^G,\psi_r'^{,-1}}(h, hA(x)) \right| |\det h|^{\mathrm{Re}(s)-m} \mathrm{d}h.$$

Using Iwasawa decomposition, note that for $k \in K_n$, $kA(x)k^{-1} = A(xk)$, the integral can be written as

$$\int_{\mathcal{P}_r(F)\backslash G_r(\mathbb{A})} \int_{[G_1]^r} \int_{K_n} \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} \left| f_{N_r^G, \psi_r'^{,-1}}\left(\left(\begin{pmatrix} h & \\ & t \end{pmatrix} k, \begin{pmatrix} h & \\ & t \end{pmatrix} A(x)k \right) \right) \right| \\ |\det h|^{\mathrm{Re}(s)-m} |\det t|^{\mathrm{Re}(s)-m} \delta_{Q_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} \mathrm{d}x\mathrm{d}k\mathrm{d}t\mathrm{d}h. \tag{4.4.4}$$

Let R_r (resp. R_r') denote the parabolic subgroup of G_{n+m} (resp. G_{n+m-1}) with Levi component $G_r \times G_1^{n+m-r}$ (resp. $G_r \times G_1^{n+m-r-1}$). Let $\begin{pmatrix} 1_n & \\ x & 1_m \end{pmatrix} = n'(x)m'(x)k'(x)$ be a measurable decomposition of $\begin{pmatrix} 1_n & \\ x & 1_m \end{pmatrix}$ under the Iwasawa decomposition $N_{R_r'}(\mathbb{A})M_{R_r'}(\mathbb{A})K_{n+m-1}$ (see §2.2.5). Then we can write $A(x)$ as $\begin{pmatrix} n'(x) & \\ & 1 \end{pmatrix} \begin{pmatrix} m'(x) & \\ & 1 \end{pmatrix} \begin{pmatrix} k'(x) & \\ & 1 \end{pmatrix} =: n(x)m(x)k(x)$, which is an Iwasawa decomposition of $A(x)$ under $G_{n+m}(\mathbb{A}) = N_{R_r}(\mathbb{A})M_{R_r}(\mathbb{A})K_{n+m}$. We also write $m(x)$ as $m(x) = \mathrm{diag}(h(x), t(x), t'(x), 1)$, where $h(x) \in \mathrm{GL}_r(\mathbb{A})$, $t(x) = \mathrm{diag}(t_1(x), \cdots, t_{n-r}(x))$ and $t'(x) = \mathrm{diag}(t_1'(x), \cdots, t_{m-1}'(x))$. The integral (4.4.4) then can be written as

$$\int_{\mathcal{P}_r(F)\backslash G_r(\mathbb{A})} \int_{[G_1]^r} \int_{K_n} \int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})} \left| (R(k, kk(x))f)_{N_r^G, \psi_r'^{,-1}}\left(\left(\begin{pmatrix} h & \\ & t \end{pmatrix}, \begin{pmatrix} h & \\ & t \end{pmatrix} m(x) \right) \right) \right| \\ |\det h|^{\mathrm{Re}(s)-m} |\det t|^{\mathrm{Re}(s)-m} \delta_{Q_r} \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} \mathrm{d}x\mathrm{d}k\mathrm{d}t\mathrm{d}h. \tag{4.4.5}$$

We will use the notation from Lemma 2.3.5. Let l denote the map

$$(u, u') \in N_{P_r} \mapsto \sum_{i=r}^{n-1} u_{i,i+1} - \sum_{j=r}^{n+m-1} u_{j,j+1} \in \mathbb{G}_a.$$

One readily checks that there exists $N_0 > 0$ such that

$$\left\| \mathrm{Ad}^* \left(\left(\begin{pmatrix} h & \\ & t \end{pmatrix}, \begin{pmatrix} h & \\ & t \end{pmatrix} m(x) \right)^{-1} l \right\|_{V_{P_r}, \mathbb{A}} \gg \|e_r h h(x)\|^{N_0} \prod_{i=1}^{n-r} \|t_i t_i(x)\|^{N_0} \prod_{i=1}^{m-1} \|t'(x)\|^{N_0}.$$

Therefore, by Lemma 2.3.5, we have

$$\left|R(k,kk(x))f_{N_r^G,\psi_r'^{,-1}}\left(\begin{pmatrix} h & \\ & t\end{pmatrix},\begin{pmatrix} h & \\ & t\end{pmatrix}m(x)\right)\right| \ll \|e_r h h(x)\|_{\mathbb{A}_r}^{-N_1}\prod_{i=1}^{n-r}\|t_i t_i(x)\|_{\mathbb{A}}^{-N_1}\prod_{i=1}^{m-1}\|t_i'(x)\|_{\mathbb{A}}^{-N_2}$$
$$\|h\|_{G_r}^{-N}\|t\|_{G_1^r}^{-N}\|tt(x)\|_{G_1^r}^{-N}\|t'(x)\|_{G_1^{m-1}}^{-N}\delta_{P_r}\left(\begin{pmatrix} h & \\ & t\end{pmatrix},\begin{pmatrix} h & \\ & t\end{pmatrix}m(x)\right)^{-cN}. \tag{4.4.6}$$

for some $c>0$ and any $N_1,N_2>0$ and $N>0$. By Lemma 2.2.2, for any $N_3>0$, we can find $N_2>0$ such that right hand side of (4.4.6) is essentially bounded by

$$\|e_r h\|^{-N_1}\prod_{i=1}^{n-r}\|t_i\|^{-N_1}\|x\|_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}^{-N_3}\|h\|_{G_r}^{-N}\|t\|_{G_1^r}^{-N}\delta_{P_r}\left(\begin{pmatrix} h & \\ & t\end{pmatrix},\begin{pmatrix} h & & \\ & t & \\ & & 1_m\end{pmatrix}\right)^{-cN}$$

Therefore the integral (4.4.5) is essentially bounded by

$$\begin{gathered}\int_{\mathcal{P}_r(F)\backslash G_r(\mathbb{A})}\int_{[G_1]^r}\int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}\|e_r h\|_{\mathbb{A}_r}^{-N_1}\prod_{i=1}^{n-r}\|t_i\|_{\mathbb{A}}^{-N_1}\|x\|_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}^{-N_3}\|h\|_{G_r}^{-N}\|t\|_{G_1^r}^{-N}\\ |\det h|^{\mathrm{Re}(s)-\alpha(N)-m}\prod_{i=1}^{n-r}|t_i|^{\mathrm{Re}(s)-\alpha_i(N)-m}\mathrm{d}x\mathrm{d}h\mathrm{d}t\end{gathered} \tag{4.4.7}$$

for any $N,N_1,N_3>0$, where $\alpha(N)$ and $\alpha_i(N)$ is as in (4.4.3). The convergence follows from the convergence of (4.4.2) when $\mathrm{Re}(s)\gg 0$.

4.4.4. *Proof of Lemma 4.1.1.* Let B_H be the upper triangular Borel subgroup of H. By Iwasawa decomposition $H(\mathbb{A})=B_H(\mathbb{A})K_n$, the integral defining $Z^{\mathrm{RS}}(s,f)$ is bounded by

$$\int_{[G_1^r]}\int_{K_n}|W'_{R(k)f}(t)||\det t|^s\delta_{B_H}(t)^{-1}\mathrm{d}t\mathrm{d}k.$$

Similar to the derivation of (4.4.2), for any $N_2>0$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_N([G])$, such that the integral is essentially bounded by

$$\|f\|\cdot\int_{[G_1^r]}\prod_{i=1}^{n}\|t_i\|^{-N_2}|t_i|^{\mathrm{Re}(s)-(n+1-2i)}\|t\|_{G_1}^{N}\mathrm{d}t.$$

Note that there exists $M>0$ such that

$$\|t\|_{G_1}\ll\max\{|t|^M,|t|^{-M}\}, \tag{4.4.8}$$

therefore the result follows from Corollary 2.2.5.

4.4.5. *Proof of Lemma 4.2.4.* Let $\begin{pmatrix} 1 & \\ x & 1_m \end{pmatrix} = n'(x)t'(x)k'(x)$ be a measurable decomposition under $G_{n+m-1}(\mathbb{A}) = N_{n+m-1}(\mathbb{A})T_{n+m-1}(\mathbb{A})K_{n+m-1}$. Then $A(x) = \begin{pmatrix} n'(x) & \\ & 1 \end{pmatrix}\begin{pmatrix} t'(x) & \\ & 1 \end{pmatrix}\begin{pmatrix} k'(x) & \\ & 1 \end{pmatrix} := n(x)t(x)k(x)$, where $t(x) = \mathrm{diag}(t_1(x), \cdots, t_n(x), t'_1(x), \cdots, t'_{m-1}(x), 1)$. Same as the derivation of (4.4.5), the integral defining $Z'_1(s,f)$ is essentially bounded by

$$\int_{[G_1]^r}\int_{K_n}\int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}\left|W''_{R(k,kk(x))f}(t,tt(x))\right||\det t|^{\mathrm{Re}(s)-m}\delta_{B_H}(t)^{-1}\mathrm{d}x\mathrm{d}k\mathrm{d}t.$$

The using the same argument for (4.4.7), for any $N_1, N_3 > 0$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_N([G])$, such that this integral is essentially bounded by

$$\|f\|\cdot\int_{[G_1]^r}\int_{\mathrm{Mat}_{(m-1\times n)}(\mathbb{A})}\prod_{i=1}^n\|t_i\|^{-N_1}\|x\|^{-N_3}_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}\|t\|^N_{G_1}\prod_{i=1}^n|t_i|^{\mathrm{Re}(s)-m-n-1+2i}\mathrm{d}x\mathrm{d}t,$$

Using (4.4.8) and Corollary 2.2.5, the result follows.

4.5. **Unfolding.** The goal of §4.5 is to prove Proposition 4.2.2 and Proposition 4.2.5.

Recall the subgroup $\mathcal{U}_0$ and the character ψ_0 on $\mathcal{U}_0(\mathbb{A})$ defined in §4.3.1. By the change of variable $h \mapsto {}^th^{-1}$, for any $f \in \mathcal{S}([G])$ we have

$$Z_{n+1}(s,f) = \int_{[H]}(R(w_{n,m})\widetilde{f})_{\{1\}\times\mathcal{U}_0,\psi_0^{-1}}(h)|\det h|^{-s}\mathrm{d}h.$$

Therefore by Corollary 4.3.3 and the absolute convergence of Z'_{n+1}, we see that

$$Z_{n+1}(s,f) = Z'_{n+1}(-s, R(w_{n,m})\widetilde{f}).$$

Therefore we are left to show:

Lemma 4.5.1. *Let $\chi \in \mathfrak{X}(G)$ be an Rankin-Selberg regular cuspidal data. Then for any $1 \le r \le n$ and $f \in \mathcal{S}_\chi([G])$,*

$$Z_r(s,f) = Z_{r+1}(s,f), \quad Z'_r(s,f) = Z'_{r+1}(s,f)$$

holds when $\mathrm{Re}(s)$ *is sufficiently large.*

Proof. For $1 \le r \le n$, recall from the introduction, we regard U_r as a subgroup of G_n. We put $U_r^G := U_r \times U_r \subset G$. We also define $U_{n+1}^G := \{1\}\times U_{n+1}$. For $1 \le r \le n+1$ Let $U_r^H := U_r^G \cap H$.

For $1 \le r \le n$, using Fourier expansion on the compact abelian group $U_r^H(\mathbb{A})U_r^G(F)\backslash U_r^G(\mathbb{A})$, we can write

$$\int_{[U_r^H]} f_{N_{r+1}^G,\psi'^{,\pm}_r}(ug)\mathrm{d}u = (f_{N_{r+1}^G,\psi'^{,\pm}_r})_{U_{r+1}^G}(g) + \sum_{\gamma\in\mathcal{P}_r(F)\backslash G_r(F)}(f_{N_{r+1}^G,\psi'^{,\pm}_r})_{U_{r+1}^G,\psi^\pm}((\gamma,\gamma)g),$$

where

$$(f_{N_{r+1}^G,\psi'^{,\pm}_r})_{U_{r+1}^G}(g) = \int_{[U_{r+1}^G]} f_{N_{r+1}^G,\psi'^{,\pm}}(ug)\mathrm{d}g,$$

and

$$\sum_{\gamma\in\mathcal{P}_r(F)\backslash G_r(F)}(f_{N^G_{r+1},\psi'_r,\pm})_{U^G_{r+1},\psi^\pm}(g)=\int_{[U^G_{r+1}]}f_{N^G_{r+1},\psi',\pm}(ug)\psi'^{,\pm}_N(u)\mathrm{d}u=f_{N^G_r,\psi'_r,\pm}(g).$$

We then formally have that $1\le r\le n$

$$Z_{r+1}(s,f)=Z_r(s,f)+F_r(s,f),\quad Z'_{r+1}(s,f)=Z'_r(s,f)+F'_r(s,f),\tag{4.5.1}$$

where

$$F_r(s,f)=\int_{G_r(F)N^H_r(\mathbb{A})\backslash H(\mathbb{A})}(f_{N^G_{r+1},\psi'_r})_{U^G_{r+1}}(h)|\det h|^s\mathrm{d}h,$$

and

$$F'_r(s,f)=\int_{G_r(F)N_{r,n}(\mathbb{A})\backslash G_n(\mathbb{A})}\int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}(f_{N^G_{r+1},\psi'^{,-1}_r})_{U^G_{r+1}}\left(h,\begin{pmatrix}1_n&&\\x&1_{m-1}&\\&&1\end{pmatrix}h\right)|\det h|^s\mathrm{d}h$$

$$=\int_{G_r(F)N_{r,n}(\mathbb{A})\backslash G_n(\mathbb{A})}\int_{\mathrm{Mat}_{(m-1)\times n}(\mathbb{A})}(f_{N^G_{r+1},\psi'^{,-1}_r})_{U^G_{r+1}}\left(h,h\begin{pmatrix}1_n&&\\x&1_{m-1}&\\&&1\end{pmatrix}\right)|\det h|^{s-m}\mathrm{d}h$$

To verify (4.5.1), we need to show that:

Lemma 4.5.2. *The integral defining $F_r(s,f)$ and $F'_r(s,f)$ are absolutely convergent when* $\mathrm{Re}(s)\gg 0$.

Assume Lemma 4.5.2 for now, it remains to show $F_r(s,f)=F'_r(s,f)=0$ for $\mathrm{Re}(s)\gg0$ and $1\le r\le n$. Note that $F_r(s,f)=0$ (for any f and ψ) implies $F'_r(s,f)=0$. We now prove that $F_r(s,f)=0$ for $\mathrm{Re}(s)\gg0$. We temporarily denote by P_r the parabolic subgroup of G with Levi component $(G_r\times G_{n-r})\times(G_r\times G_{n+m-r})$.

$$\begin{aligned}&(f_{N^G_{r+1},\psi'_r})_{U^G_{r+1}}(g)\\&=\int_{[N_{n-r}]}\int_{[N_{m+n-r}]}f_{P_r}\left(\left(\begin{pmatrix}1_r&\\&u\end{pmatrix},\begin{pmatrix}1_r&\\&u'\end{pmatrix}\right)g\right)\psi_{N_{n-r}}(u)^{-1}\psi_{N_{m+n-r}}(u')\mathrm{d}u\mathrm{d}u'.\end{aligned}$$

Let Q_r be the parabolic subgroup of G_n with Levi component $G_r\times G_{n-r}$. Using the Iwasawa decomposition $G_n(\mathbb{A})=Q_r(\mathbb{A})K_n$, we can write $F_r(s,f)$ as

$$\begin{aligned}&\int_{[G_r]}\int_{N_{n-r}(\mathbb{A})\backslash G_{n-r}(\mathbb{A})}\int_{K_n}\int_{[N_{n-r}]}\int_{[N_{n+m-r}]}\\&(R(k)f)_{P_r}\left(\begin{pmatrix}h_r&\\&uh_{n-r}\end{pmatrix},\begin{pmatrix}h_r&\\&u'\begin{pmatrix}h_{n-r}&\\&1_m\end{pmatrix}\end{pmatrix}\right)\delta_{P_r}\begin{pmatrix}h_r&\\&h_{n-r}\end{pmatrix}^{\frac{s-n+r}{2n-2r+m}}\\&|\det h_{n-r}|^{\frac{(2n+m)s+rm}{2n-2r+m}}\mathrm{d}u'\mathrm{d}u\mathrm{d}k\mathrm{d}h_{n-r}\mathrm{d}h_r.\end{aligned}\tag{4.5.2}$$

Fix N sufficiently large. By (2.3.4), for $\mathrm{Re}(s) \gg 0$, we have $[M_{P_r}] \ni m \mapsto f_{P_r}(m)\delta_{P_r}(m)^s \in L^2_{N,\chi^{M_{P_r}}}([M_{P_r}])^\infty$. We write an element of $[M_{P_r}]$ as $(h_r, x, h'_r, y) \in [G_r] \times [G_{n-r}] \times [G_r] \times [G_{n+m-r}]$. By Lemma 2.5.3, Lemma 2.5.4 and the definition of Rankin-Selberg regular, for any $(x, y) \in [G_{n-r}] \times [G_{n+m-r}]$, $(h_r, h'_r) \mapsto f_{P_r}(h_r, x, h'_r, y)\delta_{P_r}(h_r, x, h'_r, y)^s$ lies in sum of $L^2_{N,\chi}([G_r \times G_r])^\infty$, where $\chi = (\chi_r, \chi'_r)$ with $\chi_r \neq \chi_r^\vee$. Therefore, the integration of (4.5.2) over $[G_r]$ already vanishes. This finishes the proof.

It remains to prove Lemma 4.5.2. We use the notation from Lemma 2.3.5. Let l denote the map

$$(u, u') \in N_{P_r} \mapsto -\sum_{i=r+1}^{n-1} u_{i,i+1} + \sum_{i=r+1}^{m+n-1} u'_{i,i+1},$$

when $r = n-1$ or n, the first term is understood as 0. Then $(f_{N^G_{n+1},\psi'_r})_{U^G_{r+1}} = f_{N_{P_r},\psi_l}$. Using Iwasawa decomposition, the integral defining $F_r(s, f)$ is bounded by

$$\int_{[G_r]} \int_{[G_1]^{n-r}} \int_{K_n} \left| f_{N_{P_r},\psi_l}\left(\begin{pmatrix} h & \\ & t \end{pmatrix} k\right) \right| \delta_{Q_r}\begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} \mathrm{d}h\mathrm{d}t\mathrm{d}k. \tag{4.5.3}$$

Note that there exists $N_0 > 0$ such that

$$\left\| \mathrm{Ad}^* \begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} l \right\| \gg \prod_{i=1}^{n-r} \|t_i\|_{\mathbb{A}}^{N_0}.$$

By Lemma 2.3.5, the integral (4.5.3) is essentially bounded by

$$\int_{[G_r]} \int_{[G_1]^{n-r}} \prod_{i=1}^{n-r} \|t_i\|^{-N_1} \|h\|_{G_r}^{-N} \|t\|_{G_1^r}^{-N} \delta_{P_r}\begin{pmatrix} h & \\ & t \end{pmatrix}^{-cN} \delta_{Q_r}\begin{pmatrix} h & \\ & t \end{pmatrix}^{-1} |\det h|^s |\det t|^s \mathrm{d}t\mathrm{d}h,$$

whose convergence follows from the same argument of §4.4.3. The convergence of $F'_r(s, f)$ is also similar to the argument in §4.4.3 and is left to the reader. □

4.6. **A twisted version.** In §4.6, we discuss a twisted version of results in §3.1 and §4.1. We fix integers $n \geq 0$ and $m \geq 1$. Note that, in contrast with earlier part of §4, we allow $m = 1$.

4.6.1. *A twisted version.* Let $w_\ell = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in G_n$ be the longest Weyl group element. For $f \in \mathcal{S}([G])$, we define the *twisted Rankin-Selberg period* as

$$\widetilde{\mathcal{P}}_{\mathrm{RS}}(f, \Phi) := \int_{[G_n]} f_{N^G_{n+1},\psi'_{n+1}}\left(w_\ell {}^t g^{-1} w_\ell, \begin{pmatrix} g & \\ & 1_m \end{pmatrix}\right) \mathrm{d}g.$$

Let ψ_N denote the character

$$\psi_N(u, u') = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1} + \sum_{j=1}^{m+n-1} u'_{j,j+1}\right)$$

and for $f \in \mathcal{T}([G])$, its Whittaker function is defined by

$$W_f(g) = \int_{[N]} f(ug)\psi_N(u)^{-1}\mathrm{d}u.$$

For $f \in \mathcal{T}([G])$ and $s \in \mathbb{C}$, we put the twisted Zeta integral

$$\widetilde{Z}^{\mathrm{RS}}(s,f) = \int_{N_n(\mathbb{A})\backslash G_n(\mathbb{A})} W_f\left(w_\ell {}^t g^{-1} w_\ell, \begin{pmatrix} g & \\ & 1_m \end{pmatrix}\right) |\det g|^s \mathrm{d}g,$$

provided by the integral is absolutely convergent.

Let $\chi \in \mathfrak{X}(G)$. Assume that χ is represented by (M,π) where M and π are as in (4.1.1) and (4.1.2). We say that χ is *twisted Rankin-Selberg regular*, if for any $1 \le i \le s, 1 \le j \le t$, we have $\pi_{n,i} \neq \pi_{n+m,j}$. Let $\widetilde{\mathfrak{X}}_{\mathrm{RS}} \subset \mathfrak{X}(G)$ denote the set of twisted Rankin-Selberg regular cuspidal datum. We write $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$ for $\mathcal{T}_{\widetilde{\mathfrak{X}}_{\mathrm{RS}}}([G])$.

The proof of the following corollary is parallel to the proof of Corollary 3.2.4, and we omit the proof.

Corollary 4.6.1. *We have the following statements:*

(1) For $f \in \mathcal{T}([G])$, there exists $C > 0$ such that the integral defining $\widetilde{Z}^{\mathrm{RS}}(s,f)$ is convergent for $\mathrm{Re}(s) > C$ and defines a holomorphic function on $\mathcal{H}_{>C}$.

(2) The linear functional $\widetilde{\mathcal{P}}_{\mathrm{RS}}$ on $\mathcal{S}_{\widetilde{\mathrm{RS}}}([G])$ extends (uniquely) by continuity to a continuous linear functional $\widetilde{\mathcal{P}}_{\mathrm{RS}}$ on $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$.

(3) For any $f \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$, the zeta integral $Z(\cdot,f)$ extends to an entire function. And we have

$$\widetilde{\mathcal{P}}_{\mathrm{RS}}(f) = \widetilde{Z}^{\mathrm{RS}}(0,f)$$

(4) For any $s \in \mathbb{C}$, the functional $\widetilde{Z}(s,\cdot)$ on $\mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$ is continuous.

By (4.2.4), we see that

(4.6.1) The map $\widetilde{Z}^{\mathrm{RS}}(\cdot,\cdot) : \mathbb{C} \to \mathcal{T}'_{\widetilde{\mathrm{RS}}}([G]), s \mapsto (f \mapsto \widetilde{Z}^{\mathrm{RS}}(s,f))$ is holomorphic.

4.6.2. *Euler decomposition.* Let $\mathtt{S}$ be a finite set of places of F, let $\sigma = \sigma_n \boxtimes \sigma_{n+m}$ be a generic irreducible representation of $G(F_{\mathtt{S}})$. For $W \in \mathcal{W}(\sigma, \psi_{N,\mathtt{S}})$, we define local (twisted) Rankin-Selberg integral of W [JPS83] as

$$\widetilde{Z}^{\mathrm{RS}}_{\mathtt{S}}(s,W) := \int_{N_n(F_{\mathtt{S}})\backslash G_n(F_{\mathtt{S}})} W\left(w_\ell h^{-1} w_\ell, \begin{pmatrix} h & \\ & 1_m \end{pmatrix}\right) |\det h|^s \mathrm{d}h.$$

The integral defining $\widetilde{Z}^{\mathrm{RS}}_{\mathtt{S}}(s,W)$ is convergent when $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $\mathbb{C}$. Moreover, by [JPS83] and [Jac09], the quotient

$$\frac{\widetilde{Z}^{\mathrm{RS}}_{\mathtt{S}}(s,W)}{L_{\mathtt{S}}(s+\frac{m}{2}, \sigma_n^\vee \times \sigma_{n+m})}$$

is entire.

Let $P = P_n \times P_{n+m} \subset G$ be a standard parabolic subgroup and let $\pi = \pi_n \boxtimes \pi_{n+m}$ be a cuspidal automorphic representation of M_P. Assume that (M_P, π) gives a twisted Rankin-Selberg regular cuspidal data χ. Let $\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi = \Pi_n \boxtimes \Pi_{n+m}$.

For future use in §6.4, we consider a section $\varphi \in \mathcal{A}_{P,\pi_n|\cdot|^{\frac{n+m}{2}} \boxtimes \pi_{n+m}|\cdot|^{-\frac{n}{2}}}$. We write $E(\varphi)(g) = E(g, \varphi, 0)$ for the Eisenstein series of φ. Note that $E(\varphi) \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$.

Let $\mathtt{S}$ be a sufficiently large set of places of F, that we assume to contain Archimedean places as well as the places where Π, ψ or φ is ramified. We then have a decomposition $W_{E(\varphi)} = W_{E(\varphi),\mathtt{S}} W^{\mathtt{S}}_{E(\varphi)}$ such that $W^{\mathtt{S}}_{E(\varphi)}(1) = 1$ and is fixed by $K^{\mathtt{S}}$.

Note that $W_{E(\varphi),\mathtt{S}} \in \mathcal{W}(\Pi_{\mathtt{S}}, \psi_{N,\mathtt{S}})$. By the unramified computation for the Rankin-Selberg integral, we have

$$\widetilde{Z}^{\mathrm{RS}}(s, E(\varphi)) = (\Delta^{\mathtt{S},*}_{G_n})^{-1} \widetilde{Z}^{\mathrm{RS}}_{\mathtt{S}}(s, W_{E(\varphi),\mathtt{S}}) L^{\mathtt{S}}(s-n, \Pi_n^\vee \times \Pi_{n+m}) \tag{4.6.2}$$

5. Periods detecting (n,n)-Eisenstein series

5.1. **Statements of the main results.**

5.1.1. *Notations.* In §5, we will use the following notations. Let $n \geq 1$ be a fixed integer, and let $G = G_{2n}$. Let $H = \mathrm{Sp}_{2n}$, regarded as a subgroup of G. Let N denote the upper triangular unipotent subgroup of G and let $N_H := N \cap H$.

Let Q_n be the standard parabolic subgroup of G with Levi component $G_n \times G_n$. Note that $Q_n^H := Q_n \cap H$ is the Siegel parabolic subgroup of H. The Levi component of Q_n^H consists of elements of the form

$$\begin{pmatrix} J^t g^{-1} J & \\ & g \end{pmatrix}, \quad g \in G_n.$$

For $0 \leq r \leq n$, we write P_r for the standard parabolic subgroup whose Levi component is $G_1^{n-r} \times G_{2r} \times G_1^{n-r}$. Let N_r denote the unipotent radical of P_r. We denote by $P_r^H = P_r \cap H$. Note that P_0 is the Borel subgroup of G and P_0^H is the Borel subgroup of H.

Let $\mathcal{P}_{2r}$ denote the mirabolic subgroup of GL_{2r}, it consists of elements of GL_{2r} with last row $(0, \cdots, 0, 1)$. Let $\mathcal{P}_{2r}^H := \mathcal{P}_{2r} \cap \mathrm{Sp}_{2r}$. We regard Sp_{2r} as the subgroup $\begin{pmatrix} 1_{n-r} & & \\ & h & \\ & & 1_{n-r} \end{pmatrix}$ of H, where $h \in \mathrm{Sp}_{2r}$. We hence regard $\mathcal{P}_{2r}^H$ as a subgroup of H via the embedding $\mathcal{P}_{2r}^H \subset \mathrm{Sp}_{2r} \subset H$.

Let Δ denote the BZSV quadruple [MWZ24] $(G, H, \mathrm{std} \oplus \mathrm{std}^\vee, 1)$. Let ψ_n denote the degenerate character on $N(\mathbb{A})$ defined by

$$\psi_n(u) = \psi\left(\sum_{\substack{1 \leq i \leq 2n-1 \\ i \neq n}} u_{i,i+1} \right).$$

For $1 \le r \le n$, we write ψ_{N_r} for the restriction of ψ_n to $N_r(\mathbb{A})$. For $f \in \mathcal{T}([G])$, we put

$$f_{N_r,\psi}(g) := \int_{[N_r]} f(ug)\psi_{N_r}^{-1}(u)\mathrm{d}u.$$

We write K_H for the standard maximal compact subgroup of $H(\mathbb{A})$. For any semi-standard parabolic subgroup Q of H, we have the Iwasawa decomposition $H(\mathbb{A}) = Q(\mathbb{A})K_H$.

5.1.2. *The period.* For $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, we define a bilinear map $\mathcal{S}([G]) \times \mathcal{S}(\mathbb{A}_{2n}) \to \mathbb{C}$ by

$$\mathcal{P}(f,\Phi) = \int_{[H]} f(h)\Theta(h,\Phi)\mathrm{d}h.$$

By Lemma 2.3.3, the integral defining $\mathcal{P}$ is absolutely convergent and defines a continuous bilinear map on $\mathcal{S}([G]) \times \mathcal{S}(\mathbb{A}_{2n})$.

5.1.3. *Zeta integral.* For every $f \in \mathcal{T}([G])$, we associate the following degenerate Whittaker coefficient

$$V_f(g) = \int_{[N]} f(ug)\psi_n(u)^{-1}\mathrm{d}u.$$

For $f \in \mathcal{T}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$ and $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$, we set

$$Z(\lambda,f,\Phi) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} V_f(h)\Phi(e_{2n}h)e^{\langle\lambda,H_{Q_n}(h)\rangle}\mathrm{d}h$$

provided by the expression converges absolutely. Denote the unique element in Δ_{Q_n} by α. We define $s_\lambda := -\langle\lambda,\alpha^\vee\rangle$. Therefore s_λ has the property

$$\exp\left(\langle\lambda, H_{Q_n}\begin{pmatrix} J^tg^{-1}J & \\ & g\end{pmatrix}\rangle\right) = |\det g|^{s_\lambda},$$

thus inducing a linear map $\mathfrak{a}^*_{Q_n,\mathbb{C}} \to \mathbb{C}, \lambda \mapsto s_\lambda$.

The following two lemmas provide the convergence of zeta integral

Lemma 5.1.1. *Let $N \ge 0$. There exists $c_N > 0$ such that*

1. *For every $f \in \mathcal{T}_N([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the expression defining $Z(\lambda,f,\Phi)$ converges absolutely when $\mathrm{Re}(s_\lambda) > c_N$ and defines a holomorphic function of λ on the region $\mathrm{Re}(s_\lambda) > c_N$;*
2. *For every $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$ and $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$ with $\mathrm{Re}(s_\lambda) > c_N$, the functional $f \in \mathcal{T}_N([G]) \mapsto Z(\lambda,f,\Phi)$ is continuous.*

Lemma 5.1.2. *We have the following statements*

1. *For every $f \in \mathcal{S}([G])$, $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$ and $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$, the expression defining $Z(\lambda,f,\Phi)$ converges absolutely and defines an entire function in λ;*
2. *For every $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the functional $f \in \mathcal{S}([G]) \mapsto Z(\lambda,f,\Phi)$ is continuous;*

Lemma 5.1.1 and Lemma 5.1.2 will be proved in §5.2.2 and §5.2.4 respectively.

5.1.4. Δ-*regular cuspidal datum.* Let $\chi \in \mathfrak{X}(G)$ be a cuspidal data, let $\chi^{M_{Q_n}}$ be the preimage of χ in $\mathfrak{X}(M_{Q_n}) = \mathfrak{X}(\mathrm{GL}_n \times \mathrm{GL}_n)$. We say that χ is Δ-*regular*, if for any $\chi' \in \chi^{M_{Q_n}}$ is twisted Rankin-Selberg regular in the sense of §3.2.5. The reader can check that this definition is the same as the one given in (1.2.3). We remark that Δ here stands for the quadruple defined in §5.1.1. Note that any regular cuspidal data is Δ-regular.

Let $\mathfrak{X}_\Delta \subset \mathfrak{X}(G)$ denote the set of Δ-regular cuspidal data. We write $\mathcal{S}_\Delta([G])$ (resp. $\mathcal{T}_\Delta([G])$) for $\mathcal{S}_{\mathfrak{X}_\Delta}([G])$ (resp. $\mathcal{T}_{\mathfrak{X}_\Delta}([G])$).

5.1.5. *Main results.* For $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, we denote by $\Phi^\flat \in \mathcal{S}(\mathbb{A}_n)$ the restriction of Φ to $\{0\} \times \mathbb{A}_n$.

Theorem 5.1.3. *We have the following statements*

(1) For any $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the restriction of $\mathcal{P}(\cdot, \Phi)$ to $\mathcal{S}_\Delta([G])$ extends (uniquely) by continuity to a functional $\mathcal{P}^*$ on $\mathcal{T}_\Delta([G])$.

(2) For any $f \in \mathcal{T}_\Delta([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the map $\lambda \mapsto Z(\lambda, f, \Phi)$ extends to an entire function in $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$. Indeed, for any $k \in K_H$, $(R(k)f)_{Q_n}|_{[G_n \times G_n]} \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G_n \times G_n])$, and we have

$$Z(\lambda, f, \Phi) = \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + \frac{1}{2}, (R(k)f)_{Q_n}, (R(k)\Phi)^\flat)\mathrm{d}k, \tag{5.1.1}$$

here $(R(k)f)_{Q_n}$ means $(R(k)f)_{Q_n}|_{[G_n \times G_n]}$.

(3) We have

$$\mathcal{P}^*(f, \Phi) = Z(0, f, \Phi).$$

(4) The bilinear map $\mathcal{T}_\Delta([G]) \times \mathcal{S}(\mathbb{A}_{2n}) \to \mathbb{C}, (f, \Phi) \mapsto \mathcal{P}^*(f, \Phi)$ is continuous.

5.2. **Convergence of Zeta integrals.**

5.2.1. *More zeta integrals.* For $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$ and $0 \le r \le n$. We define

$$Z_r(f, \Phi) = \int_{N_r(\mathbb{A})\mathcal{P}^H_{2r}(F)\backslash H(\mathbb{A})} f_{N_r,\psi}(h)\mathrm{d}h.$$

Note that when $r = 0$, $Z_r(f, \Phi) = Z(0, f, \Phi)$.

Lemma 5.2.1. *For every $0 \le r \le n$ and $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the integral defining $Z_r(f, \Phi)$ converges absolutely.*

5.2.2. *Proof of Lemma 5.1.1.*

Proof. By the Iwasawa decomposition $H(\mathbb{A}) = P^H_0(\mathbb{A})K_H$, we need to show the existence of $c_N > 0$ such that

$$\int_{K_H} \int_{(\mathbb{A}^\times)^n} |V_f(D(a_1, \cdots, a_n)k)| \left|\Phi(a_1^{-1}e_{2n}k)\right| \delta_{P^H_0}(D(a_1, \cdots, a_n))^{-1} \prod_{i=1}^n |a_i|^{-\mathrm{Re}(s)}\mathrm{d}a_1 \cdots \mathrm{d}a_n\mathrm{d}k \tag{5.2.1}$$

when $\mathrm{Re}(s) > c_N$. Where

$$D(a_1, \cdots, a_n) = \mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}).$$

The modular function is given by

$$\delta_{P_0^H}(D(a_1, \cdots, a_n)) = \prod_{i=1}^{n} |a_i|^{2n-2i+2}.$$

We apply Lemma 2.3.5 (2), then for every $N_1 > 0$, we have

$$|V_f(D(a_1, \cdots, a_n)k)| \ll \prod_{i=1}^{n-1} \|a_i a_{i+1}^{-1}\|_{\mathbb{A}}^{-N_1} \prod_{i=1}^{n} \|a_i\|_{G_1}^{2N}$$

for $(k, a_1, \cdots, a_n) \in K_H \times (\mathbb{A}^\times)^n$. Note for every $N_1 > 0$, we have $|\Phi(a_1^{-1} e_{2n} k)| \ll \|a_1^{-1}\|_{\mathbb{A}}^{-N_1}$ for $(k, a_1) \in K_H \times \mathbb{A}^\times$. Note for every $N_2 > 0$, there exists $N_1 > 0$ such that

$$\prod_{i=1}^{n-1} \|a_i a_{i+1}^{-1}\|_{\mathbb{A}}^{-N_1} \|a_1^{-1}\|_{\mathbb{A}}^{-N_1} \ll \prod_{i=1}^{n} \|a_i^{-1}\|_{\mathbb{A}}^{-N_2}.$$

Then for every $N_2 > 0$, (5.2.1) is essentially bounded by

$$\prod_{i=1}^{n} \int_{\mathbb{A}^\times} \|a_i\|_{G_1}^{2N} \|a_i^{-1}\|_{\mathbb{A}}^{-N_2} |a_i|^{-\mathrm{Re}(s) - (2n-2i+2)} \mathrm{d}a_i \tag{5.2.2}$$

Since there exists $M > 0$ such that $\|a_i\|_{G_1} \ll \max\{|a_i|^M, |a_i|^{-M}\}$, the convergence of (5.2.2) follows from Corollary 2.2.5. □

5.2.3. *Proof of Lemma 5.2.1.*

Proof. We assume that $r > 0$, the case $r = 0$ will be covered in Lemma 5.1.2. By the Iwasawa decomposition $H(\mathbb{A}) = P_r^H(\mathbb{A}) K_H$, we need to show the convergence of

$$\begin{aligned} \int_{K_H} \int_{(\mathbb{A}^\times)^{n-r}} \int_{\mathcal{P}_{2r}^H(F) \backslash \mathrm{Sp}_{2r}(\mathbb{A})} & |f_{N_r,\psi}(D(a_1, \cdots, a_{n-r}, h)k)| \, |\Phi(a_1^{-1} e_{2n} k)| \\ & \delta_{P_r^H}(D(a_1, \cdots, a_{n-r}, h))^{-1} \mathrm{d}h \mathrm{d}a_1 \cdots \mathrm{d}a_{n-r} \mathrm{d}k, \end{aligned} \tag{5.2.3}$$

where

$$D(a_1, \cdots, a_{n-r}, h) = \mathrm{diag}(a_1, \cdots, a_{n-r}, h, a_{n-r}^{-1}, \cdots, a_1^{-1}).$$

and the modular function $\delta_{P_r^H}$ is given by

$$\delta_{P_r^H}(D(a_1, \cdots, a_{n-r}, h)) = \prod_{i=1}^{n-r} |a_i|^{2n+2-2i}.$$

We now apply Lemma 2.3.5 (1). For this, we note $\psi_{N_r} = \psi \circ l$, where $l : N_r \to \mathbb{G}_a$ sends $u \in N_r$ to $u_{1,2} + \cdots u_{n-r,n-r+1} + u_{n+r,n+r+1} + \cdots + u_{2n-1,2n}$. One can check that

$$\prod_{i=1}^{n-r-1} \|a_i a_{i+1}^{-1}\|_{\mathbb{A}} \|a_{n-r} e_{2r} h\|_{\mathbb{A}_{2r}} \ll \| \mathrm{Ad}^*(D(a_1, \cdots, a_{n-r}, h)^{-1}) l\|_{V_{P_r}, \mathbb{A}}.$$

Therefore, by 2.3.5 (1), we can find $c > 0$ such that for every $N_1, N_2 > 0$ we have

$$
\begin{aligned}
&|f_{N_r,\psi}(D(a_1, \cdots, a_{n-r}, h)k)| \\
&\ll \prod_{i=1}^{n-r-1} \|a_i a_{i+1}^{-1}\|_{\mathbb{A}}^{-N_1} \|a_{n-r} e_{2r} h\|_{\mathbb{A}_{2r}}^{-N_1} \prod_{i=1}^{n-r} \|a_i\|_{G_1}^{-2N_2} \|h\|_{G_{2r}}^{-N_2} \delta_{P_r}(D(a_1, \cdots, a_{n-r}, h))^{-cN_2}
\end{aligned}
$$

for $(k, a_1, \cdots, a_{n-r}, h) \in K_H \times (\mathbb{A}^\times)^r \times \mathrm{Sp}_{2r}(\mathbb{A})$. The modular function is

$$\delta_{P_r}(D(a_1, \cdots, a_{n-r}, h)) = \prod_{i=1}^{r} |a_i|^{4n-4i+2}.$$

On the other hand, for every $N_1 > 0$, we have

$$|(R(k)\Phi)(a_1^{-1} e_{2n})| \ll \|a_1^{-1}\|_{\mathbb{A}}^{-N_1}, \quad (k, a_1) \in K_H \times \mathbb{A}^\times.$$

One can check for every $N_3 > 0$, there exists $N_1 > 0$ such that

$$\prod_{i=1}^{n-r-1} \|a_i a_{i+1}^{-1}\|_{\mathbb{A}}^{-N_1} \|a_{n-r} e_{2r} h\|_{\mathbb{A}_{2r}}^{-N_1} \|a_1^{-1}\|_{\mathbb{A}}^{-N_1} \ll \prod_{i=1}^{n-r} \|a_i^{-1}\|_{\mathbb{A}}^{-N_3} \|e_{2r} h\|_{\mathbb{A}_{2r}}^{-N_3}.$$

Then we deduce the existence of $c > 0$ such that for every $N_3, N_2 > 0$, (5.2.3) is essentially bounded by the product of

$$\int_{\mathcal{P}_{2r}^H(F)\backslash \mathrm{Sp}_{2r}(\mathbb{A})} \|h\|_{G_{2r}}^{-N_2} \|e_{2r} h\|_{\mathbb{A}_{2r}}^{-N_3} \mathrm{d}h \tag{5.2.4}$$

and

$$\prod_{i=1}^{n-r} \int_{\mathbb{A}^\times} \|a_i^{-1}\|_{\mathbb{A}}^{-N_3} |a_i|^{-(4n-4i+2)cN_2 - (2n-2i+2)} \mathrm{d}a_i \tag{5.2.5}$$

By Lemma 2.3.3, there exists $N_0 > 0$, such that for every $N_3 \geq N_0$, we have

$$
\begin{aligned}
&\int_{\mathcal{P}_{2r}^H(F)\backslash \mathrm{Sp}_{2r}(\mathbb{A})} \|h\|_{G_{2r}}^{-N_2} \|e_{2r} h\|_{\mathbb{A}_{2r}}^{-N_3} \mathrm{d}h \\
&= \int_{[\mathrm{Sp}_{2r}]} \|h\|_{G_{2r}}^{-N_2} \left(\sum_{v \in F_{2r} \backslash \{0\}} \|vh\|_{\mathbb{A}_{2r}}^{-N_3} \right) \mathrm{d}h \ll \int_{[\mathrm{Sp}_{2r}]} \|h\|_{\mathrm{Sp}_{2r}}^{N_0 - N_2} \mathrm{d}h
\end{aligned}
$$

Therefore, by Corollary 2.2.5, the integral (5.2.5) and (5.2.4) are absolutely convergent when $N_3 \gg N_2 \gg 0$. □

5.2.4. *Proof of Lemma 5.1.2.*

Proof. By the same argument of the proof of Lemma 5.2.1, the integral defining $Z(\lambda, f, \Phi)$ is bounded by

$$\prod_{i=1}^{n} \int_{\mathbb{A}^\times} \|a_i^{-1}\|_{\mathbb{A}}^{-N_3} |a_i|^{-(4n-4i+2)cN_2 - (2n-2i+2+\mathrm{Re}(s_\lambda))},$$

which is absolutely convergent when $N_3 \gg N_2 \gg \max\{0, \mathrm{Re}(s_\lambda)\}$ by Corollary 2.2.5. □

5.3. **Unfolding.**

5.3.1. *Main result.* In §5.3, we prove the following proposition.

Proposition 5.3.1. *For any $f \in \mathcal{S}_\Delta([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, we have*

$$\mathcal{P}(f, \Phi) = Z(0, f, \Phi)$$

5.3.2. *A result of Offen.* We say a cuspidal data $\chi \in \mathfrak{X}(G)$ is *even*, if χ can be represented by (M, π), where

$$M = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{GL}_{n_k}$$

and

$$\pi = \pi_1 \boxtimes \pi_1 \boxtimes \cdots \boxtimes \pi_k \boxtimes \pi_k$$

according to this decomposition. We denote by $\mathfrak{X}_{\mathrm{even}}$ the set of even cuspidal datum, and denote its complement by $\mathfrak{X}_{\mathrm{even}}^c$.

Theorem 5.3.2 (Offen)**.** *The symplectic period is vanishing on $\mathcal{S}_{\mathfrak{X}_{\mathrm{even}}^c}([G])$. That is, for any $f \in \mathcal{S}_{\mathfrak{X}_{\mathrm{even}}^c}([G])$,*

$$\int_{[H]} f(h)\mathrm{d}h = 0.$$

Proof. It is proved in [Off06, Proposition 6.2, Theorem 6.3] (see also [LO18, §7.1]) that if $\chi \in \mathfrak{X}(G)$ is not even and $f \in \mathfrak{O}_\chi$ is a pseudo-Eisenstein series, then $\int_{[H]} f(h)\mathrm{d}h = 0$. By Lemma 2.5.2, for any $f \in \mathcal{S}_\chi([G])$, we have $\int_{[H]} f(h)\mathrm{d}h = 0$.

Finally, for any $f \in \mathcal{S}_{\mathfrak{X}_{\mathrm{even}}^c}([G])$, by Theorem 2.5.1, f can be written as $\sum_{\chi \in \mathfrak{X}_{\mathrm{even}}^c} f_\chi$, where $f_\chi \in \mathcal{S}([G])$ and the sum is absolutely convergent in $\mathcal{S}([G])$. The theorem follows. □

Corollary 5.3.3. *Let $a \geq 2b$ be integers and $\chi \in \mathfrak{X}(G_a)$. Let $P = MN$ be a standard parabolic subgroup of G_a such that G_{2b} is a factor of its Levi component M. For $\chi' \in \mathfrak{X}(M_P)$, denote by $\chi'_{2b} \in \mathfrak{X}(G_{2b})$ the component of χ' at G_{2b}. Suppose that for any $\chi' \in \chi^M$, χ'_{2b} is not even. Regard $\mathrm{Sp}_{2b} \subset G_{2b}$ as a subgroup of M, then for any $f \in \mathcal{S}_\chi([G_a])$, we have*

$$\int_{[\mathrm{Sp}_{2b}]} f_P(h)\mathrm{d}h = 0.$$

Proof. Note that δ_P is trivial on Sp_{2b}, it follows from Lemma 2.3.1 that the restriction of f_P to $[\mathrm{Sp}_{2b}]$ belongs to $\mathcal{S}([\mathrm{Sp}_{2b}])$. The integral hence converges absolutely. By Lemma 2.5.3, we have $f_P \in \mathcal{T}_\chi([G_a]_P)$. Let $\kappa \in C_c^\infty(\mathfrak{a}_P)$ be a compactly supported smooth function on $\mathfrak{a}_P$ with $\kappa(0) = 1$. By (2.3.3) and Lemma 2.5.6, we conclude that

$$(\kappa \circ H_P) \cdot f_P \in \mathcal{T}_\chi([G_a]_P) \cap \mathcal{S}([G_a]_P) = \mathcal{S}_\chi([G_a]_P).$$

By Lemma 2.5.4 and Lemma 2.5.5, the restriction of $(\kappa \circ H_P) \cdot f_P$ to $[G_{2b}]$ belongs to $\sum_{\chi' \in \chi^M} \mathcal{S}_{\chi'_{2b}}([G_{2b}])$. It follows from Theorem 5.3.2 that

$$\int_{[\mathrm{Sp}_{2b}]} f_P(h)\mathrm{d}h = \int_{[\mathrm{Sp}_{2b}]} (\kappa \circ H_P)(h) f_P(h)\mathrm{d}h = 0.$$

□

5.3.3. *Proof of Proposition 5.3.1.* Proposition 5.3.1 is implied by the following Lemma

Lemma 5.3.4. *For any $f \in \mathcal{S}_\Delta([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, we have*

$$Z_0(f,\Phi) = Z_1(f,\Phi) = \cdots = Z_n(f,\Phi) = \mathcal{P}(f,\Phi)$$

Proof. We show that $Z_r(f,\Phi) = Z_{r+1}(f,\Phi)$ for $0 \leq r \leq n-1$, the proof of $Z_n(f,\Phi) = \mathcal{P}(f,\Phi)$ is similar and is left to the reader.

Let $r \geq 1$, we denote by U_r the unipotent radical of the parabolic subgroup of GL_{2r} with Levi component $G_1 \times G_{2r-2} \times G_1$, which we regard U_r as the subgroup $\begin{pmatrix} 1_{n-r} & & \\ & u & \\ & & 1_{n-r} \end{pmatrix}, u \in U_r$ of G. Let $U_r^H := U_r \cap H$. Note that U_r^H is a normal subgroup of U_r. By an abuse of notation, we write ψ for the character $u \mapsto \psi(u_{12} + u_{2r-1,2r})$ of $U_r(\mathbb{A})$.

By Fourier inversion on the compact abelian group $U_{r+1}(\mathbb{A})/U_{r+1}(F)U_{r+1}^H(\mathbb{A})$, we have

$$\int_{[U_{r+1}^H]} f_{N_{r+1},\psi}(uh)\mathrm{d}h = (f_{N_{r+1},\psi})_{U_{r+1}} + \sum_{\gamma \in \mathcal{P}_{2r}^H(F) \backslash \mathrm{Sp}_{2r}(F)} (f_{N_{r+1},\psi})_{U_{r+1},\psi}$$

for all $h \in H(\mathbb{A})$, where we have set

$$(f_{N_{r+1},\psi})_{U_r}(g) = \int_{[U_r]} f_{N_{r+1},\psi}(ug)\mathrm{d}u,$$
$$(f_{N_{r+1},\psi})_{U_r,\psi}(g) = \int_{[U_r]} f_{N_{r+1},\psi}(ug)\psi(u)\mathrm{d}u = f_{N_r,\psi}(g).$$

Therefore, we formally have

$$Z_r(f,\Phi) = Z_{r+1}(f,\Phi) + F_r(f,\Phi) \tag{5.3.1}$$

where we have set

$$F_r(f,\Phi) = \int_{\mathrm{Sp}_{2r}(F)N_r^H(\mathbb{A}) \backslash H(\mathbb{A})} (f_{N_{r+1},\psi})_{U_r}(h)\Phi(e_{2n}h)\mathrm{d}h.$$

To verify (5.3.1), we need to show

Lemma 5.3.5. *For every $0 \leq r \leq n-1$, $f \in \mathcal{S}([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the integral defining $F_r(f,\Phi)$ converges absolutely.*

Proof of Lemma 5.3.5. By the same arguments as the proof of Lemma 5.2.1, there exists $c > 0$ such that for every $N, N_2 > 0$, the integral defining $F_r(f,\Phi)$ is essentially bounded by the product of

$$\int_{[\mathrm{Sp}_{2r}]} \|h\|_{G_{2r}}^{-N_2} \mathrm{d}h$$

and

$$\prod_{i=1}^{n-r} \int_{\mathbb{A}^\times} \|a_i\|_{G_1}^{-2N_2} \|a_i^{-1}\|_{\mathbb{A}}^{-N} |a_i|^{-(4n-4i+2)cN_2-(2n-2i+2)} \mathrm{d}a_i.$$

We can take $N \gg N_2 \gg 0$ such that these integrals converge. □

Let R_r be the standard parabolic subgroup of G with Levi component $G_{n-r} \times G_{2r} \times G_{n-r}$. Let V_k denote the upper triangular unipotent subgroup of G_k. Then

$$(f_{N_{r+1},\psi})_{U_r}(g) = \int_{[V_{n-r}]} \int_{[V_{n-r}]} f_{R_r}\left(\begin{pmatrix} u_1 & & \\ & 1_{2r} & \\ & & u_2 \end{pmatrix} g \right) \psi^{-1}(u_1)\psi^{-1}(u_2) \mathrm{d}u_1 \mathrm{d}u_2.$$

Let $R_r^H := R_r \cap H$. Using the Iwasawa decomposition $H(\mathbb{A}) = R_r^H(\mathbb{A})K_H$, we can write $F_r(f,\Phi)$ as

$$F_r(f,\Phi) = \int_{V_{n-r}(\mathbb{A})\backslash G_{n-r}(\mathbb{A})} \int_{[\mathrm{Sp}_{2r}]} \int_K \int_{[V_{n-r}]} \int_{[V_{n-r}]} f_{R_r}\left(\begin{pmatrix} u_1 g & & \\ & h & \\ & & u_2 J_{n-r} {}^t g^{-1} J_{n-r} \end{pmatrix} k \right)$$
$$\delta_{R_r^H} \begin{pmatrix} g & & \\ & h & \\ & & J_{n-r} {}^t g^{-1} J_{n-r} \end{pmatrix}^{-1} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}k \mathrm{d}h \mathrm{d}g.$$

Therefore the vanishing is implied by

$$\int_{[\mathrm{Sp}_{2r}]} f_{R_r} \begin{pmatrix} 1_{n-r} & & \\ & h & \\ & & 1_{n-r} \end{pmatrix} \mathrm{d}h \tag{5.3.2}$$

vanishes for any $f \in \mathcal{S}_\Delta([G])$. It suffices to prove that for any Δ-regular cuspidal data and any $f \in \mathcal{S}_\chi([G])$, the integral (5.3.2) vanishes. However, one can check directly that for any $\chi' = (\chi_1, \chi_2, \chi_3) \in \chi^{M_{P_r}} \subset \mathfrak{X}(G_{n-r} \times G_{2r} \times G_{n-r})$, χ_2 is not even. Therefore, the integral vanishes by Corollary 5.3.3. □

5.4. **Proof of Theorem 5.1.3.** Let $\mathfrak{X}_\Delta^{M_{Q_n}}$ denote the preimage of $\mathfrak{X}_\Delta$ in $\mathfrak{X}(M_{Q_n})$. By the definition of Δ-regularity, we have $\mathfrak{X}_\Delta^{M_{Q_n}} \subset \mathfrak{X}_{\widetilde{\mathrm{RS}}}(M_{Q_n})$ (indeed it is easy to see that this is an equality).

Let $f \in \mathcal{T}_\Delta([G])$ and $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, By the Iwasawa decomposition $H(\mathbb{A}) = Q_n^H(\mathbb{A})K_H$, when $Z(\lambda, f, \Phi)$ is absolutely convergent, we have

$$Z(\lambda, f, \Phi) = \int_{K_H} \int_{N_n(\mathbb{A})\backslash G_n(\mathbb{A})} V_{R(k)f} \begin{pmatrix} J^t g^{-1} J & \\ & g \end{pmatrix} \Phi(e_{2n} g) |\det g|^{s_\lambda + n + 1} \mathrm{d}g \mathrm{d}k. \qquad (5.4.1)$$

From the definition of the degenerate Whittaker coefficient, we have

$$V_f \begin{pmatrix} g' & \\ & g \end{pmatrix} = W_{f_{Q_n}}^{M_{Q_n}} \begin{pmatrix} g' & \\ & g \end{pmatrix}, \quad g, g' \in G_n(\mathbb{A})$$

Then we can write

$$Z(\lambda, f, \Phi) = \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + \frac{1}{2}, (R(k)\Phi)^\flat, (R(k)f)_{Q_n}) \mathrm{d}k, \quad \mathrm{Re}(s_\lambda) \gg 0.$$

Then it follows from Corollary 3.2.4 that $\widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + \frac{1}{2}, (R(k)\Phi)^\flat, (R(k)f)_{Q_n})$ extends to an entire function of s_λ. Applying Lemma 2.7.1 (2) with

$$W = \mathcal{S}(\mathbb{A}_n), \quad V = \mathcal{T}_{\widetilde{\mathrm{RS}}}([G_n \times G_n]), \quad X = K_H \times \mathcal{S}(\mathbb{A}_{2n}) \times \mathcal{T}_\Delta([G]),$$

the holomorphic map

$$s \in \mathbb{C} \mapsto \widetilde{Z}^{\mathrm{RS}}(s + n + \frac{1}{2}, \cdot, \cdot) \in \mathrm{Bil}(\mathcal{S}(\mathbb{A}_n), \mathcal{T}_{\widetilde{\mathrm{RS}}}([G_n \times G_n])),$$

and continuous maps

$$(s, k, f, \Phi) \in \mathbb{C} \times K_H \times \mathcal{S}(\mathbb{A}_{2n}) \times \mathcal{T}_\Delta([G]) \mapsto (R(k)\Phi)^\flat \in \mathcal{S}(\mathbb{A}_n),$$
$$(s, k, f, \Phi) \in \mathbb{C} \times K_H \times \mathcal{S}(\mathbb{A}_{2n}) \times \mathcal{T}_\Delta([G]) \mapsto (R(k)f)_{Q_n} \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G_n \times G_n])$$

we deduce that the map

$$(s, k, f, \Phi) \in \mathbb{C} \times K_H \times \mathcal{S}(\mathbb{A}_{2n}) \times \mathcal{T}_\Delta([G]) \mapsto \widetilde{Z}^{\mathrm{RS}}(s + n + \frac{1}{2}, (R(k)\Phi)^\flat, (R(k)f)_{Q_n}) \in \mathbb{C}$$

is continuous and holomorphic in the first variable. Then it follows from Lemma 2.7.2 that the integral

$$\int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + \frac{1}{2}, (R(k)\Phi)^\flat, (R(k)f)_{Q_n}) \mathrm{d}k$$

is holomorphic in $s \in \mathbb{C}$. Therefore $Z(\lambda, f, \Phi)$ extends to an entire function. This proves (2). Lemma 2.7.2 also implies $Z(0, f, \Phi)$ is continuous in $(f, \Phi) \in \mathcal{T}_\Delta([G]) \times \mathcal{S}(\mathbb{A}_{2n})$. Moreover by Lemma 5.1.2 and Proposition 5.3.1,

$$Z(0, f, \Phi) = \mathcal{P}(f, \Phi).$$

Therefore for $\Phi \in \mathcal{S}(\mathbb{A}_{2n})$, the $f \mapsto Z(0, f, \Phi)$ provides a continuous extension of $\mathcal{P}(\cdot, \Phi)$ to $\mathcal{T}_\Delta([G])$, this proves (1),(3) and (4).

5.5. **Periods of Δ-regular Eisenstein series.**

5.5.1. *Local zeta integral.* Fix a place v of F, let $\Pi_M = \Pi \boxtimes \Pi'$ be an irreducible generic representation of $M_{Q_n}(F_v)$. Recall the space $\mathrm{Ind}_{Q_n(F_v)}^{G(F_v)} \mathcal{W}(\Pi_M, \psi_v)$ defined in §2.6.2. For $W^M \in \mathrm{Ind}_{Q_n(F_v)}^{G(F_v)} \mathcal{W}(\Pi_M, \psi_v)$ and $\Phi \in \mathcal{S}(F_{v,2n})$ and $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, we define

$$Z_v(\lambda, W^M, \Phi) = \int_{N_H(F_v)\backslash H(F_v)} W^M(h)\Phi(e_{2n}h)e^{\langle \lambda, H_{Q_n}(h)\rangle} \mathrm{d}h,$$

provided by the integral is absolutely convergent.

Lemma 5.5.1. *(1) For any $W^M \in \mathrm{Ind}_{Q_n(F_v)}^{G(F_v)}$ and $\Phi \in \mathcal{S}(F_{v,2n})$, the integral defining $Z_v(\lambda, W^M, \Phi)$ is absolutely convergent when* $\mathrm{Re}(s_\lambda) \gg 0$ *and has a meromorphic continuation to $\mathfrak{a}_{Q_n,\mathbb{C}}^*$.*

Proof. Using the Iwasawa decomposition, we can formally write $Z_v(\lambda, W^M, \Phi)$ as

$$\begin{aligned} Z_v(\lambda, W^M, \Phi) &= \int_{K_{H,v}} \int_{N_n(F_v)\backslash G_n(F_v)} R(k)W \begin{pmatrix} J^t g^{-1} J & \\ & g \end{pmatrix} (R(k)\Phi)^\flat(e_n g)|\det g|^{s_\lambda+n+1}\mathrm{d}g\mathrm{d}k \\ &= \int_{K_{H,v}} \widetilde{Z}_v^{\mathrm{RS}}(s_\lambda + n + \frac{1}{2}, (R(k)\Phi)^\flat, R(k)W^M|_{M_{Q_n}(F_v)}). \end{aligned} \tag{5.5.1}$$

The convergence of the zeta integral hence follows from the convergence of the usual Rankin-Selberg integral [JPS83], [Jac09].

When v is non-Archimedean, by (5.5.1), $Z_v(\lambda, W^M, \Phi)$ is essentially a finite sum of twisted Rankin-Selberg integral, hence has meromorphic continuation. We now assume v is Archimedean. Let $\mathcal{O}(\mathbb{C})$ denote the entire function on $\mathbb{C}$ with the usual compact-open topology. By [Jac09, Theorem 2.3], the map

$$\mathcal{W}(|\cdot|^{\frac{n}{2}}\Pi \boxtimes |\cdot|^{-\frac{n}{2}}\Pi', \psi) \times \mathcal{S}(F_{v,n}) \to \mathcal{O}(\mathbb{C}), (W, \Phi') \mapsto \left(s \mapsto \frac{\widetilde{Z}_v^{\mathrm{RS}}(s+n+\frac{1}{2}, \Phi', W)}{L_v(s+1, \Pi^\vee \times \Pi')} \right) \tag{5.5.2}$$

is continuous. Therefore, the map

$$K_{H,v} \to \mathcal{O}(\mathbb{C}) : k \mapsto \frac{\widetilde{Z}_v^{\mathrm{RS}}(s+n+\frac{1}{2}, (R(k)\Phi)^\flat, R(k)W)}{L_{\mathtt{S}}(s+1, \Pi^\vee \times \Pi')}$$

is continuous. Combining with (5.5.1), the meromorphicity follows from Lemma 2.7.2. □

We finally remark by the same argument, Lemma 5.5.1 still holds if we replace v by a finite set $\mathtt{S}$ of places of F.

5.5.2. *Fixed points.* Let $P = M_P N_P$ be a standard parabolic subgroup, we write M_P as $G_{n_1} \times \cdots \times G_{n_k}$. Let $\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$ be a cuspidal unitary automorphic representation of M_P (central character not necessarily trivial on A_M^∞) such that the cuspidal datum χ represented by (M_P, π_0) (see §1.2.2) is Δ-regular.

We write $\mathrm{Fix}(\pi)$ for the set of permutations $\sigma : \{1,2,\cdots,k\} \to \{1,2,\cdots,k\}$ such that there exists $1 \le t \le k$ with:

(1) $n_{\sigma(1)} + \cdots + n_{\sigma(t)} = n, n_{\sigma(t+1)} + \cdots + n_{\sigma(k)} = n$.
(2) $\sigma(1) < \cdots < \sigma(t)$ and $\sigma(t+1) < \cdots < \sigma(k)$.

We also introduce the following notations

(1) P_σ the standard parabolic subgroup of G_{2n+m} with $M_{P_\sigma} = G_{n_{\sigma(1)}} \times \cdots \times G_{n_{\sigma(k)}}$,
(2) $P_{\sigma,n}$ (resp. $P'_{\sigma,n}$) the standard parabolic subgroup of G_n with Levi subgroup $G_{n_{\sigma(1)}} \times \cdots \times G_{n_{\sigma(t)}}$ (resp. $G_{n_{\sigma(t+1)}} \times \cdots \times G_{n_{\sigma(k)}}$),
(3) $\pi_\sigma = \pi_{\sigma(1)} \boxtimes \cdots \boxtimes \pi_{\sigma(k)}$, which is a cuspidal automorphic representation of M_{P_σ},
(4) $\pi_{\sigma,n} = \pi_{\sigma(1)} \boxtimes \cdots \boxtimes \pi_{\sigma(t)}$ and $\pi'_{\sigma,n} = \pi_{\sigma(t+1)} \boxtimes \cdots \boxtimes \pi_{\sigma(k)}$.
(5) $\Pi_{\sigma,n} = \mathrm{Ind}_{P_{\sigma,n}(\mathbb{A})}^{G_n(\mathbb{A})} \pi_{\sigma,n}$ and $\Pi'_{\sigma,n} = \mathrm{Ind}_{P_{\sigma,n+m}(\mathbb{A})}^{G_{n+m}(\mathbb{A})} \pi_{\sigma,n+m}$.

5.5.3. *L-functions.* Let $\sigma \in \mathrm{Fix}(\pi)$, we put

$$L(s, T_\sigma \check{X}) := L(s, \Pi_{\sigma,n} \times \Pi'^{,\vee}_{\sigma,n}) L(s, \Pi^\vee_{\sigma,n} \times \Pi'_{\sigma,n}).$$

5.5.4. *Periods of Eisenstein series.* Let $\varphi \in \Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi = \mathcal{A}_{P,\pi}$ and write $E(\varphi)(g) = E(g,\varphi,0)$ for the Eisenstein series of φ. Then $E(\varphi) \in \mathcal{T}_\Delta([G])$.

Theorem 5.5.2. *We have*

$$\mathcal{P}(E(\varphi)) = (\Delta_H^{\mathtt{S},*})^{-1} L(1,\pi,\widehat{\mathfrak{n}}_P^-)^{-1} \sum_{\sigma \in \mathrm{Fix}(\pi)} L^{\mathtt{S}}(1, T_\sigma \check{X}) L_{\mathtt{S}}(1, \pi_\sigma, \widehat{\mathfrak{n}}_{P_\sigma}^-) Z_{\mathtt{S}}(\lambda, \Phi_{\mathtt{S}}, \Omega_{\mathtt{S}}^{M_{Q_n}}(N_{\pi,\mathtt{S}}(\sigma) W^M_{\varphi,\mathtt{S}})).$$

Recall the L-function $L(s,\pi,\widehat{\mathfrak{n}}_P^-$ defined in (2.4.3).

Proof. By the constant term formula for Eisenstein series, we have

$$(R(k)E(\varphi))_{Q_n} = \sum_{w \in W(P;Q_n)} E^{Q_n}(M(w)R(k)\varphi), \quad k \in K_H.$$

By Theorem 5.1.3, we can write

$$\begin{aligned} \mathcal{P}^*(E(\varphi),\Phi) &= \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(n+\frac{1}{2}, (R(k)E(\varphi))_{Q_n}, (R(k)\Phi)^\flat) \mathrm{d}k \\ &= \sum_{w \in W(P;Q_n)} \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(n+\frac{1}{2}, E^{Q_n}(M(w)R(k)\varphi), (R(k)\Phi)^\flat) \mathrm{d}k. \end{aligned} \tag{5.5.3}$$

where the second equality holds because for each $w \in W(P;Q_n)$ and each $k \in K_H$, $E^{Q_n}(M(w)R(k)\varphi) \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([M_{Q_n}])$.

Assume that there exists $1 \le i < j \le k$ such that $\pi_i^\vee = \pi_j$, then by the computation of Fourier coefficient of Eisenstein series [Sha81, §4], the Whittaker function of $E^{Q_n}(M(w)R(k)\varphi)|_{[M_Q]}$ vanishes for any $k \in K_H$, therefore $\mathcal{P}^*(E(\varphi),\Phi)$ vanishes. Therefore, from now on, we assume that $\pi_i^\vee \ne \pi_j$ for any $i \ne j$. In particular, for any finite subset $\mathtt{S}$ of places of F, the partial L-function $L^{\mathtt{S}}(s,\pi_i^\vee \times \pi_j)$ is regular (and non-vanishing) at $s=1$.

Let $\mathtt{S}$ be a sufficiently large finite set of places of F, which we assume to contain Archimedean places as well as the places where Π or ψ is ramified. We also assume φ is fixed by $K^{\mathtt{S}}$ and Φ can be written as $\Phi = \Phi_{\mathtt{S}}\Phi^{\mathtt{S}}$, where $\Phi^{\mathtt{S}}$ is the characteristic function of $\mathcal{O}^{\mathtt{S}}_{F,2n}$ and $\Phi_{\mathtt{S}} \in \mathcal{S}(F_{\mathtt{S},2n})$.

Note that there is a bijection between and $W(P;Q_n)$ and $\mathrm{Fix}(\pi)$, where each w corresponds to the σ such that $wM_Pw^{-1} = M_{P,\sigma}$. In the following, we fix an arbitrary $w \in W(P;Q_n)$, and corresponding $\sigma \in \mathrm{Fix}(\pi)$. It's clear that under this correspondence, one can identify the representation $w\pi$ of wM_Pw^{-1} with the representation π_σ of $M_{P,\sigma}$.

Note that the restriction of $E^{Q_n}(M(w)R(k)\varphi)$ to $[M_{Q_n}]$ belongs to $|\cdot|^{\frac{n}{2}}\Pi_{\sigma,n} \boxtimes |\cdot|^{-\frac{n}{2}}\Pi'_{\sigma,n}$, then it follows from (3.2.5) and (5.5.1) that

$$\int_{K_H} \widetilde{Z}^{\mathrm{RS}}(n+\frac{1}{2}, (R(k)\Phi)^\flat, E^{Q_n}(M(w)R(k)\varphi))\mathrm{d}k = (\Delta_H^{\mathtt{S},*})^{-1}L^{\mathtt{S}}(1, \Pi_{\sigma,n}^\vee \times \Pi'_{\sigma,n})Z_{\mathtt{S}}(0, W^{M_{Q_n}}_{M(w)E(\varphi),\mathtt{S}}, \Phi_{\mathtt{S}}). \tag{5.5.4}$$

By (2.6.3) and (2.6.4), we have

$$\begin{aligned} W^{M_{Q_n}}_{E^{Q_n}(M(w)\varphi),\mathtt{S}} &= \frac{1}{L^{\mathtt{S}}(1,\pi_{\sigma,n},\widehat{\mathfrak{n}}^-_{P_{\sigma,n}})L^{\mathtt{S}}(1,\pi'_{\sigma,n},\widehat{\mathfrak{n}}^-_{P'_{\sigma,n}})}\frac{L(1,\pi_\sigma,\widehat{\mathfrak{n}}^-_{P_\sigma})}{L(1,\pi,\widehat{\mathfrak{n}}^-_P)}\Omega^{M_{Q_n}}_{\mathtt{S}}(N_{\pi,\mathtt{S}}(w)W^{M_P}_{\varphi,\mathtt{S}}) \\ &= \frac{L^{\mathtt{S}}(1,\Pi_{\sigma,n}\times\Pi'^{,\vee}_{\sigma,n})}{L(1,\pi,\widehat{\mathfrak{n}}^-_P)}L_{\mathtt{S}}(1,\pi_\sigma,\widehat{\mathfrak{n}}^-_{P_\sigma})\Omega^{Q_n}_{\mathtt{S}}(N_{\pi,\mathtt{S}}(w)W^{M_P}_{\varphi,\mathtt{S}}) \end{aligned}$$

Therefore we can write the left hand side of (5.5.4) as

$$(\Delta_H^{\mathtt{S},*})^{-1}\frac{L^{\mathtt{S}}(1,\Pi^\vee_{\sigma,n}\times\Pi'_{\sigma,n})L^{\mathtt{S}}(1,\Pi^\vee_{\sigma,n}\times\Pi'_{\sigma,n})}{L(1,\pi,\widehat{\mathfrak{n}}^-_P)}L_{\mathtt{S}}(1,\pi_\sigma,\widehat{\mathfrak{n}}^-_{P_\sigma})Z_{\mathtt{S}}(0,\Omega^{Q_n}_{\mathtt{S}}(N_{\pi,\mathtt{S}}(w)W^{M_P}_{\varphi,\mathtt{S}}),\Phi).$$

This finishes the proof $\square$

6. Periods detecting $(n, n+m)$-Eisenstein series

6.1. **Statement of the main results.**

6.1.1. *Notations.* In §6, fix integers $n \geq 0$ and $m \geq 1$. Let $G = G_{2n+m}$ and let $H = \mathrm{Sp}_{2n}$. We regard H as the subgroup $\begin{pmatrix} h & \\ & 1 \end{pmatrix}$, $h \in H$ of G. We will study period related to the quadruple $\Delta := \Delta_{n,m} = (G, H, 0, \iota_{n,m})$, where $\iota_{n,m} : \mathrm{SL}_2 \to G$ is the representation $\mathbf{1}^{n+1} \oplus \mathrm{Sym}^{m-1}$ of SL_2.

Let $N = N_{2n+m}$ denote the upper triangular unipotent subgroup of G and let $N^H := N \cap H$.

For $0 \leq r \leq n$, let $P_r := P_r^{n,m}$ be the parabolic subgroup of of G whose Levi component is isomorphic to $G_1^{n-r} \times G_{2r} \times G_1^{n+m-r}$. Let $P_r^H := P_r \cap H$, it is a parabolic subgroup of H whose unipotent radical is $N_r^H := N_r \cap H$. The Levi component of P_r^H is $\mathrm{Sp}_{2r} \times G_1^{n-r}$.

Let $\mathcal{P}_{2r}$ denote the mirabolic subgroup of GL_{2r}, it consists of elements of GL_{2r} with last row $(0, \cdots, 0, 1)$. Let $\mathcal{P}_{2r}^H := \mathcal{P}_{2r} \cap \mathrm{Sp}_{2r}$. We regard Sp_{2r} as the subgroup $\begin{pmatrix} 1_r & & \\ & h & \\ & & 1_r \end{pmatrix}$ of H, where $h \in \mathrm{Sp}_{2r}$. We hence regard $\mathcal{P}_{2r}^H$ as a subgroup of H via the embedding $\mathcal{P}_{2r}^H \subset \mathrm{Sp}_{2r} \subset H$.

Let ψ_n denote the degenerate character

$$N(\mathbb{A}) \ni u \mapsto \psi\left(\sum_{\substack{1 \le i \le n+m-1 \\ i \neq n}} u_{i,i+1}\right)$$

of $N(\mathbb{A})$ which is trivial on $N(F)$.

We also denote by N_{n+1} the unipotent radical of the parabolic P_{n+1} of G whose Levi component is $G_{n+1} \times G_1^{n+m-1}$.

For $1 \le r \le n+1$, We write ψ_{N_r} for the restriction of ψ_n to $N_r(\mathbb{A})$. For $f \in \mathcal{T}([G])$, we put

$$f_{N_r,\psi}(g) := \int_{[N_r]} f(ug)\psi_{N_r}^{-1}(u)\mathrm{d}u.$$

6.1.2. *The period.* For $f \in \mathcal{S}([G])$, we define the period $\mathcal{P} := \mathcal{P}_\Delta$ on $\mathcal{S}([G])$ by

$$\mathcal{P}(f) = \int_{[H]} f_{N_{n+1},\psi}(h)\mathrm{d}h.$$

By Lemma 2.5.3, the integral

$$\int_{[H]} \int_{[N_{n+1}]} |f(nh)|\mathrm{d}n\mathrm{d}h$$

is absolutely convergent. Hence the integral defining $\mathcal{P}(f)$ is absolutely convergent.

6.1.3. *Zeta integral.* For $f \in \mathcal{T}([G])$, we associate the *degenerate Whittaker coefficient*

$$V_f(g) = \int_{[N]} f(ug)\psi_n^{-1}(u)\mathrm{d}u.$$

Note that $V_f(g) = f_{N_0,\psi}(g)$. Let Q_n denote the parabolic subgroup of G of type $(n, n+m)$. For $f \in \mathcal{T}([G])$, and for $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, we set

$$Z(\lambda, f) = \int_{N_H(\mathbb{A}) \backslash H(\mathbb{A})} V_f(h) e^{\langle \lambda, H_{Q_n}(h) \rangle}\mathrm{d}h,$$

provided by the integral is absolutely convergent. Note that $Z(\lambda, f)$ only depends on $s_\lambda := \langle \lambda, \alpha^\vee \rangle \in \mathbb{C}$.

Lemma 6.1.1. *We have the following statements:*

1. *for any $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, the integral defining $Z(\lambda, f)$ is absolutely convergent, and it defines an entire function on $\mathfrak{a}_{Q_n,\mathbb{C}}^*$,*
2. *for any $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, the map $f \mapsto Z(\lambda, f)$ is continuous on $\mathcal{S}([G])$.*

Lemma 6.1.2. *Let $N > 0$, then there exists $c_N > 0$ such that*

(1) *The integral defining $Z(\lambda, f)$ is absolutely convergent when $f \in \mathcal{T}_N([G])$ and $\mathrm{Re}(s_\lambda) > c_N$, and defines a holomorphic function of λ on the region $\mathrm{Re}(s_\lambda) > c_N$.*
(2) *For fix λ such that $\mathrm{Re}(s_\lambda) > c_N$. The map $\mathcal{T}_N([G]) \ni f \mapsto Z(\lambda, f) \in \mathbb{C}$ is continuous.*

The proof of the two lemmas is parallel to proofs given in §5.2, so we leave it to the readers.

6.1.4. *Δ-regular cuspidal datum.* Let $\chi \in \mathfrak{X}(G)$ be a cuspidal data, let $\chi^{M_{Q_n}}$ be the preimage of χ in $\mathfrak{X}(M_{Q_n}) = \mathfrak{X}(\mathrm{GL}_n \times \mathrm{GL}_{n+m})$. We say that χ is Δ-*regular*, if for any $\chi' \in \chi^{M_{Q_n}}$ is twisted Rankin-Selberg regular in the sense of 4.6.1. We remark that Δ here stands for the quadruple defined in §6.1.1. Note that any regular cuspidal data is Δ-regular.

Let $\mathfrak{X}_\Delta \subset \mathfrak{X}(G)$ denote the set of Δ-regular cuspidal data. We write $\mathcal{S}_\Delta([G])$ (resp. $\mathcal{T}_\Delta([G])$) for $\mathcal{S}_{\mathfrak{X}_\Delta}([G])$ (resp. $\mathcal{T}_{\mathfrak{X}_\Delta}([G])$).

6.1.5. *Main results.*

Theorem 6.1.3. *We have the following statements*

(1) *The restriction of $\mathcal{P}$ to $\mathcal{S}_\Delta([G])$ extends (uniquely) by continuity to a functional $\mathcal{P}^*$ on $\mathcal{T}_\Delta([G])$.*
(2) *For any $f \in \mathcal{T}_\Delta([G])$, the map $\lambda \mapsto Z(\lambda, f)$ extends to an entire function in $\lambda \in \mathfrak{a}^*_{Q_n,\mathbb{C}}$. Indeed, for any $k \in K_H$, $(R(k)f)_{Q_n}|_{[G_n \times G_{n+m}]} \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G_n \times G_{n+m}])$, and we have*
$$Z(\lambda, f) = \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + 1, (R(k)f)_{Q_n}) \mathrm{d}k, \tag{6.1.1}$$
here $(R(k)f)_{Q_n}$ means $(R(k)f)_{Q_n}|_{[G_n \times G_{n+m}]}$
(3) *We have*
$$\mathcal{P}^*(f) = Z(0, f).$$

The proof of the Proposition will be given in §6.3.

6.2. **Unfolding.** In §6.2, we show the following result:

Proposition 6.2.1. *For any $f \in \mathcal{S}_\Delta([G])$, we have*
$$\mathcal{P}(f) = Z(0, f).$$

6.2.1. *More zeta integrals.* For $f \in \mathcal{S}([G])$, we put
$$Z_r(f) = \int_{N_r^H(\mathbb{A})\mathcal{P}_{2r}^H(F)\backslash H(\mathbb{A})} f_{N_r.\psi_r}(h) \mathrm{d}h.$$
Note that $Z_0(f) = Z(0, f)$.

Proposition 6.2.2. *For any $f \in \mathcal{S}([G])$, the integral defining $Z_r(f)$ is absolutely convergent.*

Proof. The proof of the proposition follows the same line of the proof of Lemma 5.2.1, and we omit the proof. □

6.2.2. Proposition 6.2.1 will directly follow from the following lemma.

Lemma 6.2.3. *For any $f \in \mathcal{S}_\Delta([G])$, we have*

$$Z_0(f) = Z_1(f) = \cdots = Z_n(f) = \mathcal{P}(f).$$

Proof. It suffices to prove $Z_n(f) = \mathcal{P}(f)$ and $Z_r(f) = Z_{r+1}(f)$ for any $0 \le r \le n-1$. We prove the latter, and the former one follows from a similar argument.

Let $r \ge 1$, we denote by U_r the unipotent radical of the parabolic subgroup of GL_{2r} with Levi component $G_1 \times G_{2r-2} \times G_1$, which we regard U_r as the subgroup $\begin{pmatrix} 1_{n-r} & & \\ & u & \\ & & 1_{n+m-r} \end{pmatrix}$, $u \in U_r$ of G. Let $U_r^H := U_r \cap H$. By an abuse of notation, we write ψ for the character $u \mapsto \psi(u_{12} + u_{2r-1,2r})$ of $U_r(\mathbb{A})$.

Using Fourier analysis on the compact abelian group $U_{r+1}(\mathbb{A})/U_{r+1}^H(\mathbb{A})U_{r+1}(F)$, we can write

$$\int_{[U_{r+1}^H]} f_{N_{r+1},\psi}(uh)\mathrm{d}h = (f_{N_{r+1},\psi})_{U_{r+1}} + \sum_{\gamma \in \mathcal{P}_{2r}^H(F)\backslash H_{2r}(F)} (f_{N_{r+1},\psi})_{U_{r+1},\psi}.$$

where

$$(f_{N_{r+1},\psi})_{U_r}(g) = \int_{[U_r]} f_{N_{r+1},\psi}(ug)\mathrm{d}u,$$
$$(f_{N_{r+1},\psi})_{U_r,\psi}(g) = \int_{[U_r]} f_{N_{r+1},\psi}(ug)\psi(u)\mathrm{d}u = f_{N_r,\psi}(g).$$

Therefore, we formally have

$$Z_{r+1}(f) = Z_r(f) + F_r(f), \tag{6.2.1}$$

where

$$F_r(f) = \int_{\mathrm{Sp}_{2r}(F)N_r(\mathbb{A})\backslash H(\mathbb{A})} (f_{N_{r+1},\psi})_{U_r}(h)\mathrm{d}h.$$

To verify (6.2.1), we need to show that the integral defining $F_r(f)$ is absolutely convergent. The proof follows the same line of the proof of Lemma 5.3.5, and we omit the proof. Therefore, we are reduced to show that for $0 \le r \le n-1$, we have $F_r(f) = 0$.

Let R_r denote the parabolic subgroup of G with Levi component $G_{n-r} \times G_{2r} \times G_{n+m-r}$. V_k denote the upper triangular unipotent subgroup of G_k. Write $R_r^H := R_r \cap H$. Using Iwasawa decomposition $H(\mathbb{A}) = R_r^H(\mathbb{A})K_H$, we can write the integral defining $F_r(f)$ as

$$F_r(f) = \int_{V_{n-r}(\mathbb{A})\backslash G_{n-r}(\mathbb{A})} \int_{[\mathrm{Sp}_{2r}]} \int_K \int_{[V_{n-r}]} \int_{[V_{n+m-r}]} f_{R_r}\left(\cdot \begin{pmatrix} u_1 g. & & \\ & h & \\ & & u_2 \begin{pmatrix} J_{n-r}{}^t g^{-1} J_{n-r} & \\ & 1_m \end{pmatrix} \end{pmatrix} k \right)$$

$$\delta_{R_r^H}^{-1} \begin{pmatrix} .g & & \\ & h & \\ & & J_{n-r}{}^t g^{-1} J_{n-r} \end{pmatrix}^{-1} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}k \mathrm{d}h \mathrm{d}g.$$

Therefore, the vanishing of $F_r(f)$ is implied by

$$\int_{[\mathrm{Sp}_{2r}]} f_{R_r} \begin{pmatrix} 1_{n-r} & & \\ & h & \\ & & 1_{n+m-r} \end{pmatrix} \mathrm{d}h \tag{6.2.2}$$

vanishes for any $f \in \mathcal{S}_\Delta([G])$. This follows from Corollary 5.3.3. $\square$

6.3. **Proof of Theorem 6.1.3.** Let $\mathfrak{X}_\Delta^{M_{Q_n}}$ denote the preimage of $\mathfrak{X}_\Delta$ in $\mathfrak{X}(M_{Q_n})$, then we have $\mathfrak{X}_\Delta^{M_{Q_n}} \subset \mathfrak{X}_{\widetilde{\mathrm{RS}}}(M_{Q_n})$

Therefore, it follows from Lemma 2.5.3, for any $k \in K_H$ and $f \in \mathcal{T}_\Delta([G])$, we have $R(k)f|_{[M_{Q_n}]} \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([M^{Q_n}])$.

Using Iwasawa decomposition as in (5.4.1), we see that for any $f \in \mathcal{T}_\Delta([G])$, the equality

$$Z(\lambda, f) = \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + 1, ((R(k)f)_{Q_n})|_{[M_{Q_n}]}) \mathrm{d}k \tag{6.3.1}$$

holds when $\mathrm{Re}(s_\lambda) \gg 1$.

By Corollary 4.6.1, for $f' \in \mathcal{T}_{\widetilde{\mathrm{RS}}}([G])$, $\widetilde{Z}^{\mathrm{RS}}(s, f')$ has holomorphic continuation to $s \in \mathbb{C}$ and is continuous in f'. Therefore, $\widetilde{Z}^{\mathrm{RS}}(s_\lambda + n + 1, (R(k)f)_{Q_n})|_{[M_{Q_n}]}$ is defined for any $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$. We argue as in §5.4 that the right hand side of (6.3.1) is holomorphic in λ, and for any $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, $f \mapsto Z(\lambda, f)$ is continuous in $f \in \mathcal{T}_\Delta([G])$. Therefore (2) is proved.

By Proposition 6.2.1, the functional $Z(0, \cdot)$ on $\mathcal{T}_\Delta([G])$ coincides with $\mathcal{P}$ on the dense subspace $\mathcal{S}_\Delta([G])$. Therefore $f \mapsto Z(0, f)$ provides an extension of $\mathcal{P}$ to $\mathcal{T}_\Delta([G])$, (1) and (3) then follow.

6.4. Periods of Δ-regular Eisenstein series.

6.4.1. *Local zeta integral.* Fix a place v of F, let $\Pi_M = \Pi_n \boxtimes \Pi_{n+m}$ be an irreducible generic representation of $M_{Q_n}(F_v)$. For $W^M \in \mathrm{Ind}_{Q_n(F_v)}^{G(F_v)} \mathcal{W}(\Pi_M, \psi_v)$ and $\lambda \in \mathfrak{a}_{Q_n,\mathbb{C}}^*$, we define

$$Z_v(\lambda, W^M) = \int_{N_H(F_v)\backslash H(F_v)} W^M(h) e^{\langle \lambda, H_{Q_n}(h) \rangle} \mathrm{d}h,$$

provided by the integral is absolutely convergent.

Lemma 6.4.1. *For any* $W^M \in \mathrm{Ind}_{Q_n(F_v)}^{G(F_v)}$, *the integral defining* $Z_v(\lambda, W^M)$ *is absolutely convergent when* $\mathrm{Re}(s_\lambda) \gg 0$ *and has a meromorphic continuation to* $\mathfrak{a}^*_{Q_n,\mathbb{C}}$.

We omit the proof which is parallel to the proof of Lemma 5.5.1. By Iwasawa decomposition, we can write

$$Z_v(\lambda, W^M) = \int_{K_H} \widetilde{Z}_v^{\mathrm{RS}}(s_\lambda + n + 1, (R(k)W^M)|_{M_{Q_n}(F_v)}) \tag{6.4.1}$$

6.4.2. *Fixed points.* Let $P = M_P N_P$ be a standard parabolic subgroup, we write M_P as $G_{n_1} \times \cdots \times G_{n_k}$. Let $\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$ be a cuspidal unitary automorphic representation of M_P (central character not necessarily trivial on A_M^∞) such that the cuspidal datum χ represented by (M_P, π_0) is Δ-regular.

Recall the set $\mathrm{Fix}(\pi)$ defined in 1.2.3.

(1) P_σ the standard parabolic subgroup of G_{2n+m} with $M_{P_\sigma} = G_{n_{\sigma(1)}} \times \cdots \times G_{n_{\sigma(k)}}$,
(2) $P_{\sigma,n}$ (resp. $P_{\sigma,n+m}$) the standard parabolic subgroup of G_n (resp. G_{n+m}) with Levi subgroup $G_{n_{\sigma(1)}} \times \cdots \times G_{n_{\sigma(t)}}$ (resp. $G_{n_{\sigma(t+1)}} \times \cdots \times G_{n_{\sigma(k)}}$),
(3) $\pi_\sigma = \pi_{\sigma(1)} \boxtimes \cdots \boxtimes \pi_{\sigma(k)}$, which is a cuspidal automorphic representation of M_{P_σ},
(4) $\pi_{\sigma,n} = \pi_{\sigma(1)} \boxtimes \cdots \boxtimes \pi_{\sigma(t)}$ and $\pi_{\sigma,n+m} = \pi_{\sigma(t+1)} \boxtimes \cdots \boxtimes \pi_{\sigma(k)}$.
(5) $\Pi_{\sigma,n} = \mathrm{Ind}_{P_{\sigma,n}(\mathbb{A})}^{G_n(\mathbb{A})} \pi_{\sigma,n}$ and $\Pi_{\sigma,n+m} = \mathrm{Ind}_{P_{\sigma,n+m}(\mathbb{A})}^{G_{n+m}(\mathbb{A})} \pi_{\sigma,n+m}$.

For $\sigma \in \mathrm{Fix}(\pi)$, we put

$$L(s, T_\sigma \check{X}) := L(s, \Pi_{\sigma,n}^\vee \times \Pi_{\sigma,n+m}) L(s, \Pi_{\sigma,n} \times \Pi_{\sigma,n+m}^\vee).$$

6.4.3. *Periods of Eisenstein series.* Let $\varphi \in \Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G_{2n}(\mathbb{A})} \pi = \mathcal{A}_{P,\pi}$ and write $E(\varphi)(g) = E(g, \varphi, 0)$ for the Eisenstein series of φ. Note that $E(\varphi) \in \mathcal{T}_\Delta([G])$.

Theorem 6.4.2. *Let* $\mathtt{S}$ *be a sufficiently large finite set of places of* F, *that contains Archimedean places and the places where* Π *or* ψ *is ramified. We also assume that* φ *is fixed by* $K^{\mathtt{S}}$, *and we decompose* $W_\varphi^{M_P}$ *as* $W_\varphi^{M_P} = W_{\varphi,\mathtt{S}}^{M_P} W_\varphi^{M_P,\mathtt{S}}$. *Then period* $\mathcal{P}^*(E(\varphi))$ *is equal to*

$$(\Delta_H^{\mathtt{S},*})^{-1} L(1, \pi, \widehat{\mathfrak{n}}_P^-)^{-1} \sum_{\sigma \in \mathrm{Fix}(\pi)} L^{\mathtt{S}}(1, \Pi, T_\sigma \check{X}) L_{\mathtt{S}}(1, \pi_\sigma, \widehat{\mathfrak{n}}_{P_\sigma}^-) Z_{\mathtt{S}}(0, \Omega_{\mathtt{S}}^{M_{Q_n}}(N_{\pi,\mathtt{S}}(\sigma) W_{\varphi,\mathtt{S}}^{M_P}). \tag{6.4.2}$$

Proof. The proof is parallel to the proof of Theorem 5.5.2, so we will be brief.

By the constant term formula for Eisenstein series andTheorem 6.1.3, we can write

$$\mathcal{P}^*(E(\varphi)) = \sum_{w \in W(P;Q_n)} \int_{K_H} \widetilde{Z}^{\mathrm{RS}}(n+1, E^{Q_n}(M(w)R(k)\varphi)) \mathrm{d}k. \tag{6.4.3}$$

If there exists $1 \le i < j \le k$ such that $\pi_i^\vee = \pi_j$, then both side of (6.4.2) is 0, therefore from now on we assume that $\pi_i \neq \pi_j$ for $i \neq j$.

Let $\mathtt{S}$ be a sufficiently large finite set of places of F, which we assume to contain Archimedean places as well as the places where Π or ψ is ramified. We also assume φ is fixed by $K^{\mathtt{S}}$. Note that there is a bijection between $W(P; Q_n)$ and $\mathrm{Fix}(\pi)$, where each w corresponds to the

σ such that $wM_Pw^{-1} = M_{P,\sigma}$. In the following, we fix an arbitrary $w \in W(P; Q_n)$, and corresponding $\sigma \in \mathrm{Fix}(\pi)$.

Note that the restriction of $E^{Q_n}(M(w)R(k)\varphi)$ to $[M_{Q_n}]$ belongs to $|\cdot|^{\frac{n+m}{2}}\Pi_{\sigma,n}\boxtimes|\cdot|^{-\frac{n}{2}}\Pi_{\sigma,n+m}$, then it follows from (4.6.2) and (6.4.1) that

$$\int_{K_H} \widetilde{Z}^{\mathrm{RS}}(n+1, E^{Q_n}(M(w)R(k)\varphi))\mathrm{d}k = (\Delta_H^{\mathtt{S},*})^{-1} L^{\mathtt{S}}(1, \Pi_{\sigma,n}^\vee \times \Pi_{\sigma,n+m}) Z_{\mathtt{S}}(0, W^{M_{Q_n}}_{M(w)E(\varphi),\mathtt{S}}). \tag{6.4.4}$$

By (2.6.3) and (2.6.4), we have

$$W^{M_{Q_n}}_{E^{Q_n}(M(w)\varphi),\mathtt{S}} = \frac{L^{\mathtt{S}}(1, \Pi_{\sigma,n} \times \Pi^\vee_{\sigma,n+m})}{L(1, \pi, \widehat{\mathfrak{n}}_P^-)} L_{\mathtt{S}}(1, \pi_\sigma, \widehat{\mathfrak{n}}_{P_\sigma}^-)\Omega^{Q_n}_{\mathtt{S}}(N_{\pi,\mathtt{S}}(w)W^{M_P}_{\varphi,\mathtt{S}}) \tag{6.4.5}$$

The theorem then follows from (6.4.3), (6.4.4), and (6.4.5). $\square$

7. Truncation operator and the regularized period

7.1. **Notations.** Let $H = \mathrm{Sp}_{2n}$. We fix an upper triangular Borel subgroup P_0' of H, let $\mathfrak{a}_{P_0'} := \mathfrak{a}_0'$, and $\Delta_0' = \Delta_{P_0'}$

Let $G = G_{2n+1}$. For a semi-standard parabolic subgroup $P \subset G$, let $\mathfrak{a}_P^+$ be the subset of $X \in \mathfrak{a}_P$ such that $\langle X, \alpha\rangle > 0$ for any $\alpha \in \Delta_P$.

For any semi-standard parabolic subgroups $P \subset Q$, let $\widehat{\tau}_P^Q$ be the usual characteristic function of a cone on $\mathfrak{a}_P$ defined in [Art78, §5].

7.2. **The case** $m = 1$**.** The case $m = 1$ is taking the Sp_{2n} period of an automorphic form on GL_{2n+1}. In the work [Zyd19] of Zydor, he defined a regularized period of an automorphic form on a reductive group over any reductive subgroup.

Let $G = G_{2n+1}$ and $H = \mathrm{Sp}_{2n}$. Zydor's regularization was written down explicitly in [LWX25, §3.2] in this case, which we also briefly review here.

Let $\mathcal{F}'$ be the set of standard parabolic subgroups of H. For each $P' \in \mathcal{F}'$, there is a unique semi-standard parabolic subgroup of G such that $\mathfrak{a}_P^+ \cap \mathfrak{a}_{P'}^+ \neq \varnothing$. If we write $P' = P(\lambda)$ via the dynamical method, where λ is a cocharacter of H. Then P can also be characterized as $P = P(\lambda^G)$, where λ^G denotes the corresponding cocharacter of G. In the following, we will also denote by a standard parabolic subgroup of H with a letter with a $'$, and the corresponding parabolic subgroup of G will be denoted by the same letter without $'$.

Let $f \in \mathcal{T}([G])$, we define

$$\Lambda^T f(h) = \sum_{P' \in \mathcal{F}'} \varepsilon_{P'} \sum_{\gamma \in P'(F)\backslash H(F)} \widehat{\tau}_{P'}(H_{P'}(\gamma h) - T_{P'}) f_P(\gamma h).$$

By [Zyd19, Theorem 3.9] (see also [LWX25, Theorem 3.2.2]), when T is sufficiently positive, $\Lambda^T f \in \mathcal{S}^0([H])$, moreover, the map $f \in \mathcal{T}([G]) \mapsto \Lambda^T f \in \mathcal{S}^0([H])$ is continuous. For such T, we define

$$\mathcal{P}^T(f) := \int_{[H]} \Lambda^T f(h) \mathrm{d}h.$$

More generally, for $Q' \in \mathcal{F}'$ and $f \in \mathcal{T}(Q(F)\backslash G(\mathbb{A}))$ (see [BLX24, §4.3] for a definition), we define $\Lambda^{T,Q'} f$ by

$$\Lambda^{T,Q'} f(h) = \sum_{\substack{P' \in \mathcal{F}' \\ P' \subset Q'}} \varepsilon_{P'}^{Q'} \sum_{\gamma \in P'(F)\backslash Q'(F)} \widehat{\tau}_{P'}^{Q'}(H_{P'}(\gamma h) - T_{P'}) f_P(\gamma h).$$

We can similarly show that $\Lambda^{T,Q'} \in \mathcal{S}^0([H]^1_{Q'})$ and the map $f \in \mathcal{T}(Q(F)\backslash G(\mathbb{A})) \to \Lambda^{T,Q'} f \in \mathcal{S}^0([H]^1_{Q'})$ is continuous.

There is also a variant of truncation operator for Levi subgroup. Let $Q' \in \mathcal{F}'$ and $f \in \mathcal{T}([M_{Q'}])$ and $T \in \mathfrak{a}_0'$, we define

$$\Lambda^{T,M_{Q'}} f(h) = \sum_{\substack{P' \in \mathcal{F}' \\ P' \subset Q'}} \varepsilon_{P'}^{Q'} \sum_{\gamma \in (M_{Q'} \cap P'(F))\backslash M_{Q'}(F)} \widehat{\tau}_{P'}^{Q'}(H_{P'}(\gamma h) - T_{P'}) f_{P \cap M_Q}(\gamma h).$$

Since $\delta_P^{Q,-1}$ is bounded on $\{h \in M_{Q'}(\mathbb{A}) \mid \widehat{\tau}_{P'}^{Q'}(H_{P'}(h) - T) = 1\}$. By Lemma 2.3.1, for $f \in \mathcal{S}([M_Q])$, the integral

$$\int_{[M_{Q'}]_{P \cap M_{Q'}}} \widehat{\tau}_{P'}^{Q'}(H_{P'}(h) - T_{P'}) f_{P \cap M_Q}(h)$$

is absolutely convergent. As a consequence,

(7.2.1) For $f \in \mathcal{S}([M_{Q'}])$ we have

$$\int_{[M_{Q'}]} \Lambda^{T,M_{Q'}} f(h) \mathrm{d}h = \sum_{\substack{P' \in \mathcal{F}' \\ P' \subset Q'}} \varepsilon_{P'}^{Q'} \int_{[M_{Q'}]_{P' \cap M_{Q'}}} \widehat{\tau}_{P'}^{Q'}(H_{P'}(h) - T_{P'}) f_{P \cap M_Q}(h) \mathrm{d}h.$$

Similarly,

(7.2.2) For $f \in \mathcal{S}([M_{Q'}])$ we have

$$\int_{[M_{Q'}]^1} \Lambda^{T,M_{Q'}} f(h) \mathrm{d}h = \sum_{\substack{P' \in \mathcal{F}' \\ P' \subset Q'}} \varepsilon_{P'}^{Q'} \int_{[M_{Q'}]^1_{P' \cap M_{Q'}}} \widehat{\tau}_{P'}^{Q'}(H_{P'}(h) - T_{P'}) f_{P \cap M_Q}(h) \mathrm{d}h.$$

We say that $T \in \mathfrak{a}_0' \to \infty$ if $\langle T, \alpha \rangle \to \infty$ for any $\alpha \in \Delta_0'$. Therefore, when $T \to \infty$, $\tau_{P'}(H_P(h) - T_{P'}) \to 0$ for any $h \in H(\mathbb{A})$. Therefore, by the dominated convergence theorem, we see that

(7.2.3) For any $f \in \mathcal{S}([G])$, we have

$$\lim_{T \to \infty} \mathcal{P}^T(f) = \mathcal{P}(f).$$

Let M be a Levi subgroup of G. We write $\mathfrak{X}_\Delta^M$ for the preimage of $\mathfrak{X}_\Delta$ in $\mathfrak{X}(M)$. Let $\mathcal{S}_\Delta([M]) := \mathcal{S}_{\mathfrak{X}_\Delta^M}([G])$.

Lemma 7.2.1. *Let Q' be a proper parabolic subgroup of H. Then for any $f \in \mathcal{S}_\Delta([M_Q])$, we have*

$$\int_{[M_{Q'}]^1} f(h)\mathrm{d}h = 0.$$

Proof. Let $\chi \in \mathfrak{X}_\Delta^{M_Q}$, and $f \in \mathcal{S}_\chi([M_Q])$, it suffices to show $\int_{[M_{Q'}]^1} f = 0$.

Assume that $M_{Q'} = G_{n_1} \times \cdots \times G_{n_k} \times \mathrm{Sp}_{2r}$, then $\int_{[M'_Q]^1}$ is the product of

- The "twisted diagnal period" on $\mathcal{S}([G_{n_i} \times G_{n_i}])$

$$f \mapsto \int_{[G_{n_i}]^1} f(g, w_\ell {}^t g^{-1} w_\ell)\mathrm{d}g,$$

  where w_ℓ denotes the longest Weyl element as usual.
- The symplectic period on $\mathcal{S}([G_{2r}])$:

$$f \mapsto \int_{[\mathrm{Sp}_{2r}]} f(h)\mathrm{d}h.$$

Then from Theorem 5.3.2 and the definition of $\mathfrak{X}_\Delta^{M_Q}$, it is easy to see that at least one of the integral above is vanishing. □

Combining Lemma 7.2.1 and (7.2.2), we see that

(7.2.4) Let $f \in \mathcal{S}_\Delta([M_{Q'}])$ and $T \in \mathfrak{a}'_0$ be sufficiently positive. Then

$$\int_{[M_{Q'}]^1} \Lambda^{T, M_{Q'}} f(h)\mathrm{d}h = 0.$$

Proposition 7.2.2. *Let $f \in \mathcal{T}_\Delta([G])$, then $\mathcal{P}^T(f)$ is a constant of T, and this constant is equal to $\mathcal{P}^*(f)$ in Theorem 6.1.3.*

Proof. For $P' \in \mathcal{F}'$, let $\Gamma'_{P'}$ be the function on $\mathfrak{a}_{P'} \times \mathfrak{a}_{P'}$ defined in [Art81, §2]. The function $\Gamma'_{P'}$ is compactly supported in the first variable when the second variable stays in a compact subset and

$$\widetilde{\tau}_{P'}(H - X) = \sum_{\substack{P' \in \mathcal{F}' \\ P' \subset Q'}} \varepsilon_{Q'} \widehat{\tau}_{P'}^{Q'}(H) \Gamma'_{Q'}(H, X).$$

From this, for $f \in \mathcal{T}([G])$, $T, T' \in \mathfrak{a}'_0$ sufficiently positive, we can write

$$\Lambda^{T+T'} f(h) = \sum_{Q' \in \mathcal{F}'} \sum_{\delta \in Q'(F) \backslash H(F)} \Gamma'_{Q'}(H_{Q'}(\delta h) - T_{Q'}, T'_{Q'}) \Lambda^{T, Q'} f(\delta h).$$

Therefore

$$\mathcal{P}^{T+T'}(f) - \mathcal{P}^T(f) = \sum_{\substack{Q' \in \mathcal{F}' \\ Q' \neq H}} \int_{[H]_{Q'}} \Gamma'_{Q'}(H_{Q'}(h) - T_{Q'}, T'_{Q'}) \Lambda^{T, Q'} f(h)\mathrm{d}h.$$

It remains to show that for any $f \in \mathcal{T}_\Delta([G])$ and any $Q' \neq H \in \mathcal{F}'$, the integral

$$\int_{[H]^1_{Q'}} \Lambda^{T,Q'} f(h) \mathrm{d}h$$

vanishes. As $\mathcal{S}_\Delta([G])$ is dense in $\mathcal{T}_\Delta([G])$ and the integral above is continuous in f. Therefore it suffices to show the vanishing for $f \in \mathcal{S}_\Delta([G])$. However, this directly follows from (7.2.1). The final statement then follows from (7.2.3) $\square$

Appendix A. Computation of the fixed point and the tangent space

In the Appendix, we do an exercise in linear algebra. We show that, under the hypothetical Langlands correspondence, the fixed points of the L-parameter and the L-function $L(T_x \check{X})$ coincide with the concrete description in §1.2.3 and §5.5.2. In particular, the analogue of Theorem 1.2.1 for function field matches with the Conjecture 1.1.1.

A.1. **The global Langlands correspondence.** We will assume the following properties of the hypothetical global Langlands correspondence:

(1) There exists a locally compact topological group $\mathcal{L}_F$, such that there is a bijection of isomorphism classes:

$$\{n\text{-dim continuous irreducible rep. of } \mathcal{L}_F\} \longleftrightarrow \{\text{cuspidal automorphic rep. of } G_n(\mathbb{A})\}.$$

For a cuspidal automorphic representation π of G_n, we write ϕ_π the corresponding representation $\mathcal{L}_F \to \mathrm{GL}_n(\mathbb{C})$, and called it the *$L$-parameter* of π.

(2) Let $P = MN$ be a standard parabolic subgroup of G_n. Let π be a unitary cuspidal automorphic representation of M. By the correspondence above, we have an L-parameter $\mathcal{L}_F \to \check{M}$ of π.

Let $\Pi = \mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})} \pi$, realized as Eisenstein series on $G_n(\mathbb{A})$. Then the L-parameter of Π is given by (or defined to be) $\mathcal{L}_F \to \check{M} \to \mathrm{GL}_n(\mathbb{C})$.

A.2. **The fixed points.** Let Γ be a group. Let $n > 0, m \geq 0$ be integers. Assume that we have a $2n+m$-dimensional semisimple complex representation ϕ of Γ. Γ acts on $\check{X} := \mathrm{GL}_{2n+m}(\mathbb{C}) / \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n+m}(\mathbb{C})$ via the representation $\Gamma \to \mathrm{GL}_{2n+m}(\mathbb{C})$ composes with the natural action of $\mathrm{GL}_{2n+m}(\mathbb{C})$ on $\check{X}$. We write $\mathrm{Fix}(\phi)$ for the set of fixed point of Γ on $\check{X}$.

Assume that ϕ decomposes as

$$\phi = \bigoplus_{i=1}^{k} \phi_i,$$

where ϕ_i is an n_i-dimensional irreducible representation of Γ. We remark that ϕ_i and ϕ_j may be isomorphic for $i \neq j$.

We may identify $\check{X}$ with the set of pairs (V, W) where V, W are subspaces of $\mathbb{C}^{2n+m}$ with $\dim V = n, \dim W = n+m$ and $\mathbb{C}^{2n+m} = V \oplus W$, where the action of $\mathrm{GL}_{2n+m}(\mathbb{C})$ is given by $g \cdot (V, W) = (gV, gW)$. The set $\mathrm{Fix}(\phi)$ then corresponds to decompose the representation into a direct sum of an n-dimensional invariant subspace and an $n+m$-dimensional invariant subspace.

Considering the following condition:

(A.2.1) For any subset $I \subset \{1, 2, \cdots, k\}$ such that $\sum_{i \in I} n_i = n$, we have $\phi_i \ncong \phi_j$ for any $i \in I$ and $j \notin I$.

If the condition (A.2.1) does not hold. Take a subset I such that $\sum_{i \in I} n_i = n$ and $\phi_i \cong \phi_j$ for some $i \in I$ and $j \notin I$. Then the subrepresentation $\phi_i \oplus \phi_j$ has infinitely many decomposition into irreducible representation. Take any such decomposition $\phi_i \oplus \phi_j = \rho \oplus \rho'$, then the pair

$$\left(\sum_{\substack{s \in I \\ s \neq i}} \phi_s + \rho, \sum_{\substack{t \notin I \\ t \neq j}} \phi_t + \rho' \right)$$

is a fixed point. Therefore there are infinitely many fixed points.

Conversely, if the condition (A.2.1) holds. Let $\mathbb{C}^{2n+m} = V \oplus W$ be a decomposition of Γ-representation. Then (A.2.1) implies that each isotypic part of ϕ must completely lie inside V or W. Since isotypic part is canonical, then $\mathrm{Fix}(\phi)$ is finite. To conclude, we have shown the following lemma

Lemma A.2.1. *The set* $\mathrm{Fix}(\phi)$ *is discrete (in the Zariski topology, so equivalent to finite) if and only if the condition* (A.2.1) *holds.*

From the discussion above, it is easy to see the following

Lemma A.2.2. *When* $\mathrm{Fix}(\phi)$ *is finite, the set* $\mathrm{Fix}(\phi)$ *is in bijection with the set*

$$\left\{ I \subset \{1, 2, \cdots, k\} \mid \sum_{i \in I} n_i = n \right\}$$

Finally, the following lemma describe the representation given by the tangent space of fixed point.

Lemma A.2.3. *When* $x = (V, W) \in \mathrm{Fix}(\phi)$. *Then the representation of* Γ *at* $T_x \check{X}$ *is isomorphic to* $V^\vee \otimes W \oplus V \otimes W^\vee$.

Proof. It suffices to show that for $x = (V, W) \in \check{X}$, then as a $\mathrm{GL}(V) \times \mathrm{GL}(W)$ representation, $T_x \check{X} \cong V^\vee \otimes W \oplus V \otimes W^\vee = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V)$.

Let $\mathbb{C}[\varepsilon]$ be the ring of dual number. Then $T_x \check{X}$ can be identified with the pair of free $\mathbb{C}[\varepsilon]$-submodule $(\mathbb{V}, \mathbb{W})$ of $\mathbb{C}[\varepsilon]^{2n+m}$ such that $\mathbb{C}[\varepsilon]^{2n+m} = \mathbb{V} \oplus \mathbb{W}$ and $\mathbb{V} \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} = V$, $\mathbb{W} \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} = W$. Then it is direct to check that all such $(\mathbb{V}, \mathbb{W})$ is of the form $(V + \varepsilon V + \varepsilon SV, W + \varepsilon W + \varepsilon TW)$ for $(S, T) \in \mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$. This finishes the proof. $\square$

References

[Art78] James G Arthur. "A trace formula for reductive groups I terms associated to classes in G (Q)". In: (1978).

[Art81] James Arthur. "The trace formula in invariant form". In: *Ann. of Math. (2)* 114.1 (1981), pp. 1–74.

[BCZ22] Raphaël Beuzart-Plessis, Pierre-Henri Chaudouard, and Michał Zydor. "The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case". In: *Publ. Math. Inst. Hautes Études Sci.* 135 (2022), pp. 183–336.

[Ber88] Joseph N. Bernstein. "On the support of Plancherel measure". In: *Journal of Geometry and Physics* 5.4 (1988), pp. 663–710.

[Beu21] Raphaël Beuzart-Plessis. "Comparison of local relative characters and the Ichino-Ikeda conjecture for unitary groups". In: *J. Inst. Math. Jussieu* 20.6 (2021), pp. 1803–1854.

[BL24] Joseph Bernstein and Erez Lapid. "On the meromorphic continuation of Eisenstein series". In: *Journal of the American Mathematical Society* 37.1 (2024), pp. 187–234.

[BLX24] Paul Boisseau, Weixiao Lu, and Hang Xue. "The global Gan-Gross-Prasad conjecture for Fourier-Jacobi periods on unitary groups". In: (2024). arXiv: `2404.07342 [math.RT]`.

[Boi25] Paul Boisseau. *On some spectral aspects of relative trace formulae and the Gan–Gross–Prasad conjectures, PhD thesis.* 2025.

[BSV24] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh. *Relative Langlands Duality.* 2024. arXiv: `2409.04677 [math.RT]`.

[CV24] Eric Y. Chen and Akshay Venkatesh. *Some Singular Examples of Relative Langlands Duality.* 2024. arXiv: `2405.18212 [math.NT]`.

[FJ93] Solomon Friedberg and Hervé Jacquet. "Linear periods". In: *J. Reine Angew. Math.* 443 (1993), pp. 91–139.

[Gro97] Benedict H Gross. "On the motive of a reductive group". In: *Inventiones mathematicae* 130.2 (1997), pp. 287–314.

[IT13] Atsushi Ichino and Nicolas Templier. "On the Voronoi formula for GL (n)". In: *American Journal of Mathematics* 135.1 (2013), pp. 65–101.

[Jac09] Hervé Jacquet. "Archimedean Rankin-Selberg integrals". In: *Automorphic forms and L-functions II. Local aspects.* Vol. 489. Contemp. Math. Amer. Math. Soc., Providence, RI, 2009, pp. 57–172.

[JPS83] Hervé Jacquet, Ilya I Piatetskii-Shapiro, and Joseph A Shalika. "Rankin-selberg convolutions". In: *American journal of mathematics* 105.2 (1983), pp. 367–464.

[Lap08] Erez M Lapid. "A remark on Eisenstein series". In: *Eisenstein series and applications.* Springer, 2008, pp. 239–249.

[LO18] Erez Moshe Lapid and Omer Offen. “On the distinguished spectrum of Sp_{2n} with respect to $\mathrm{Sp}_n \times \mathrm{Sp}_n$”. In: *Kyoto Journal of Mathematics* 58.1 (2018), pp. 101–171.

[LS87] R. P. Langlands and D. Shelstad. “On the definition of transfer factors”. In: *Mathematische Annalen* 278.1 (Mar. 1987), pp. 219–271.

[LWX25] Weixiao Lu, Zeyu Wang, and Guodong Xi. *On the relative Langlands duality for* $\mathrm{Sp}_{2n} \setminus \mathrm{GL}_{2n+1}$ *(with an appendix by Zeyu Wang)*. 2025. arXiv: `2504.18774 [math.RT]`.

[MS11] SD Miller and W Schmid. “A general Voronoi summation formula for $\mathrm{GL}(n, \mathbb{Z})$, Geometry and Analysis. No. 2”. In: *Adv. Lect. Math.(ALM)* 18 (2011), p. 173.

[MW89] Colette Mœglin and J-L Waldspurger. “Le spectre résiduel de $\mathrm{GL}(n)$”. In: *Annales scientifiques de l’École normale supérieure*. Vol. 22. 4. 1989, pp. 605–674.

[MW95] C Moeglin and J. L. Waldspurger. *Spectral Decomposition and Eisenstein Series: A Paraphrase of the Scriptures*. Cambridge Tracts in Mathematics. Cambridge University Press, 1995.

[MWZ24] Zhengyu Mao, Chen Wan, and Lei Zhang. *BZSV Duality for Some Strongly Tempered Spherical Varieties*. 2024. arXiv: `2310.17837 [math.NT]`.

[Off06] Omer Offen. “Residual spectrum of GL_{2n} distinguished by the symplectic group”. In: (2006).

[Sak13] Yiannis Sakellaridis. “Spherical functions on spherical varieties”. In: *American Journal of Mathematics* 135.5 (2013), pp. 1291–1381.

[Sha81] Freydoon Shahidi. “On Certain L-Functions”. In: *American Journal of Mathematics* 103.2 (1981), pp. 297–355.

[Wan24] Xiaolei Wan. *The Sakellaridis-Venkatesh Conjecture for* $U(2)\backslash SO_{2,3}$. 2024. arXiv: `1910.09026 [math.RT]`.

[Zyd19] Michał Zydor. *Periods of automorphic forms over reductive subgroups*. 2019. arXiv: `1903.01697 [math.NT]`.