0.1 Local Langlands for $\mathrm{GL}(n)$

References are [?, ?, ?, ?, ?].

Notation (0.1.0.1).

- Let $p \neq \ell \in \texttt{Prime}$.
- Let $K \in p\text{-}\texttt{NField}$.
- Let $\psi \neq 1 \in K^\vee$.
- Let $(K_r, \mathcal{O}_r, \varpi_K, \kappa_r)/K$ be an unramified extension of degree r.

⌟

1 Local Langlands for p-adic $\mathrm{GL}(n)$

Thm. (0.1.1.1) [LLC for $\mathrm{GL}(n)$, Hasse(30)/Tunnell(78)/Kutzko(80)/Harris-Taylor[?]/ Henniart(84, 86, 88, 93, 00)[?]]. There exists a unique collection of bijections $\{\mathrm{rec}_n\}_{n\in\mathbb{Z}_+}$ between sets:

$$\mathrm{rec}_n : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \xrightarrow{\cong} \mathfrak{wd}^n_{\varphi\text{-ss}}(W_K)$$

satisfying the following properties:

1. For a quasi-character χ of $K^\times$, $\mathrm{rec}_1(\chi) = \chi \circ \mathrm{Art}_K^{-1}$**??**.
2. For a quasi-character χ of $K^\times$ and $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$,
$$\mathrm{rec}_n(\pi(\chi)) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi).$$
3. For any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ with central character ω,
$$\det(\mathrm{rec}_n(\pi)) = \mathrm{rec}_1(\omega).$$
4. For any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$, $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^*$.
5. For any two $\pi_1 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_1;K)), \pi_2 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_2;K))$,
$$L(\pi_1 \times \pi_2; s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2); s), \quad \epsilon(\pi_1 \times \pi_2, \psi; s) = \epsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2); s)\text{????}.$$

⌟

Prop. (0.1.1.2) [Reduction to Supercuspidal Representations, Bernstein-Zelevinski[?]]. By Bernstein-Zelevinsky classification**??**, it suffices to construct rec_n for irreducible cuspidal representations, then for any $\pi = Q(\Delta_1, \ldots, \Delta_r) \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$, where $\Delta_i = \Delta_i(\pi_i, m_i)$, we can define

$$\mathrm{rec}_n(\pi) = \bigoplus_{i=1}^r \mathrm{rec}_{n_i}(\pi_i) \otimes \mathrm{Sp}(m_i) \in \mathfrak{wd}^n_{\varphi\text{-ss}}(W_K)\text{??}.$$

⌟

Proof: Check this satisfies the properties 1-5.? □

Cor. (0.1.1.3). If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ and $\mathrm{rec}_n(\pi) = (\rho_0, V, N)$, then ρ_0 only depends on the supercuspidal support of π**??**. ⌟

Thm. (0.1.1.4) [Uniqueness of the Correspondence, Henniart[?]]. For $n \in \mathbb{Z}_{\geq 2}$, suppose $\pi, \pi' \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$ s.t. for any $r \in [n-1]_+$ and $\tau \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(r;K))$, we have

$$\epsilon(\pi \times \tau, \psi; s) = \epsilon(\pi' \times \tau, \psi; s)$$

then $\pi \cong \pi'$. ⌟

Proof: Cf.[?]. □

Thm. (0.1.1.5) [Injectivity of the Correspondence]. Any set of maps $\mathrm{rec}_n : \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K)) \to \mathfrak{wd}^n_{\varphi\text{-ss}}(W_K)$ satisfying the properties listed in(0.1.1.1) must be injective. ⌟

Proof: By the definition of Rankin-Selberg L-factors**??** and**??**,

$$\mathrm{rec}(\pi) = \mathrm{rec}(\pi') \iff \mathrm{ord}_{s=0} L(\mathrm{rec}(\pi)^\vee \otimes \mathrm{rec}(\pi'); s) < 0 \iff L(\pi^\vee \times \pi'; s) < 0 \iff \pi = \pi'.$$

□

Thm. (0.1.1.6) [Surjectivity, Numerical Local Langlands Correspondence, Henniart[?]]. For any $n \in \mathbb{Z}_+, d \in \mathbb{N}$ and quasi-character χ of $K^\times$, the subset of $\mathrm{Irr}^{\mathrm{adm,cusp}}(\mathrm{GL}(n;K))$ of representations with conductor d and central character ω is finite, and the subset of $\mathrm{Irr}^n_{\varphi\text{-ss}}(\mathrm{WD}_K)$ of representations with conductor d and determinant ω is finite. Moreover, this two sets have the same cardinality.

In particular, any map $\mathrm{rec}_n : \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K)) \to \mathrm{Irr}^n_{\varphi\text{-ss}}(\mathrm{WD}_K)$ satisfying properties listed in(0.1.1.1) must be surjective. ⌟

Proof: □

2 Proof

The proof reduces to constructing all the Galois representations corresponding to supercuspidal representations. The method of construction of $\sigma(\pi)$ for supercuspidal π is to globalize and construct the local representations from global representations coming from cohomology of Shimura varieties. The difficulty lies in showing that all the globalization gives the same representation, and satisfies the functional equations.

The strategy is as follows:

- Choose a CM field F an $w \in \Sigma_F^{\mathrm{fin}}$ s.t. $F_w \cong K$.
- Look at the cohomology of projective $(n-1)$-dimensional PEL-type Shimura pro-variety $X = \varprojlim_{m \in \mathbb{Z}_+} X_m$ associated to the reductive group $G/\mathbb{Q}$ s.t.

$$G \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^\times \times \mathrm{GL}(n;K) \times \mathbf{G}_{D_p}$$

- For each $\xi \in \mathrm{Irr}^{\mathrm{fd}}_{\mathbb{Q}}(G)$, we can associate a lisse $\overline{\mathbb{Q}_\ell}$-sheaf L_ξ on X. The cohomologies

$$H^i(X, L_\xi) = \varinjlim_{U \in G(\mathbf{A}_{F,f})} \mathrm{H}^i_{\text{ét}}((X_U)_{\overline{F}}, L_\xi)$$

are infinite-dimensional $\overline{\mathbb{Q}_\ell}$-representations of $G(\mathbf{A}_{F,f}) \times \mathrm{Gal}_F$. We can choose ξ s.t. $H^i(X, L_\xi) = 0$ for $i \neq n-1$.

- There is a map
$$R_\xi : \mathrm{Irr}^{\mathrm{adm}}(G(\mathbf{A}_f)) \to \mathrm{Rep}^{\mathrm{fd}}(W_K) : \Pi \mapsto \mathrm{Hom}_{G(\mathbf{A}_{F,f})}(\Pi, H^{n-1}(X, L_\xi)).$$
- The decomposition $G(\mathbf{A}_f) = \mathbb{Q}_p^\times \times \mathrm{GL}(n;K) \times$(other terms) gives a decomposition $\Pi = \Pi_0 \otimes \Pi_w \otimes \Pi^w$. For any $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$, we can find such a Π s.t. Π_w is an unramified twist of Π, Π_0 is unramified and $R_\xi(\Pi) \neq 0$.
- Choose an integral model $\widetilde{X}_m$ of $X_m \otimes F_w$ and consider the completions $R_{K,n,m}$ of the local rings of a certain stratum of the special fibre of $\widetilde{X}_m$, which is the deformation space of the unique 1-dimensional ϖ_K-divisible formal $\mathcal{O}_K$-module $\Sigma_{K,n,m}$ of $\mathcal{O}_K$-height n with Drinfeld level-m structure over $\overline{\kappa}$. These spaces have canonical vanishing cycle sheaves $\psi^i_{n,m}$ and the limits of their cohomologies ψ^i_n are endowed with an admissible action of the subgroup $A_{K,n} \subset \mathrm{GL}(n;K) \times D^\times_{K,1/n} \times W_K$ consisting of elements (γ, δ, σ) s.t.
$$|\det \gamma|^{-1}.|\,\mathrm{Nmrd}(\delta)|.|\,\mathrm{Art}_K^{-1}\sigma| = 1$$
- (Harris-Taylor Construction Theorem B)Local Jacquet-Langlands theory associates to each $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$ a $\rho = \text{J-L}(\pi^\vee) \in \mathrm{Irr}^{\mathrm{adm}}(D^*_{K,1/n})$. Then there exists $r_\ell(\pi) \in \mathrm{Rep}^n_{\overline{\mathbb{Q}}_\ell}(W_K)$ s.t.
$$[\pi \otimes r_\ell(\pi)]^{\mathrm{ss}} = \sum_{i=0}^{n-1} (-1)^{n-1-i} [\mathrm{Hom}_{D^*_{K,1/n}}(\text{J-L}(\pi^\vee), \Psi^i_n)]^{\mathrm{ss}}.$$
and
$$n[R_\xi(\Pi) \otimes \chi(\Pi_0 \circ \mathrm{Nm}_{K/\mathbb{Q}_p})]^{\mathrm{ss}} \in \mathbb{Z}[r(\pi)]^{\mathrm{ss}}.$$
- To show the last assertion, one gives a description of $H^{n-1}(X, L_\xi)^{\mathbb{Z}_p^\times}$ in which $[\mathrm{Hom}_{D^*_{K,1/n}}(\text{J-L}(\pi^\vee), \Psi^i_n)]$ occurs(Theorem D).
- Define $\mathrm{rec}_n(\pi) = r_\ell(\pi^\vee(\frac{1-n}{2}))$.

Scholze's Proof

Slogan: the local Langlands correspondence for $\mathrm{GL}(n;K)$ is realized in the cohomology of the moduli space of 1-dimensional p-divisible groups of height n.

Thm. (0.1.2.1) [Test Function Characterization of Local Langlands, Scholze[?]].

(a) For each $n \in \mathbb{Z}_+$, there is a unique map
$$\sigma_n : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \to \mathfrak{wd}^n_{\psi\text{-ss}}(W_K)$$
s.t. for any $\tau \in W_K$ and any "cut-off" function $h \in C_c^\infty(\mathrm{GL}(n;K))$,
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma_n(\pi))\,\mathrm{tr}(h|\pi)(0.1.4.4),$$
Write $\mathrm{rec}'(\pi) = \sigma_n(\pi)(\frac{1-n}{2})$.

(b) If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ is a constituent of $\pi_1 \times \ldots \times \pi_r$, then $\mathrm{rec}'(\pi) = \mathrm{rec}'(\pi_1) \oplus \ldots \oplus \mathrm{rec}'(\pi_r)$.

(c) rec' induces a bijection between $\mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$ and $\mathrm{Irr}^n_{\varphi\text{-ss}}(W_K)$.

(d) rec' is compatible with twists, central characters, duals, and L- and ϵ-factors of pairs, hence $\mathrm{rec}' = \mathrm{rec}$ as in(0.1.1.1).

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Proof: (a) and (b) follow from(0.1.2.2) and(0.1.2.4).

(c) uses computation of I_K-invariant nearby cycles for simple Shimura varieties. This computation leads to a direct proof of the bijective correspondence for supercuspidal representations, without using the numerical local Langlands correspondence.

Finally, for the proof of (d): By Brauer induction and linearity, it suffices to assume that π is induced from characters. It suffices to show that: For any $\pi_1 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_1;K)), \pi_2 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_2;K))$, there exists $F \in \mathtt{NField}$ with $w \in \Sigma_F^{\mathrm{fin}}$ s.t. $K \cong F_w$, and two potentially Abelian $\pi_i \in \mathrm{Irr}^{\mathrm{auto}}(\mathrm{GL}(n_i)/F), \pi_2 \in \mathrm{Irr}^{\mathrm{auto}}(\mathrm{GL}(n_2)/F)$ s.t. $(\Pi_i)_w$ is an unramified twist of π_i. Cf. proof of[**?**]VII.2.10**?**.

Then the compatibility follows from Henniart's method of twisting with highly ramified characters, cf. Corollary 2.4 of [**?**].**?** □

Prop. (0.1.2.2) [Dévissage for Constructing Galois Representations]. For $n \in \mathbb{Z}_+$, suppose (a) and (b) of(0.1.2.1) hold for all $n' < n$ and the following hold:

1. If $\pi = \pi_1 \times \ldots \times \pi_r \in \mathrm{Rep}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ where $\pi_i \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_i;K))$, then
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}\left(\tau\Big| \bigoplus_{1\le i\le r} \sigma(\pi_i)(\frac{n-n_i}{2})\right)\mathrm{tr}(h|\pi).$$
2. For $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ that is either essentially square-integrable or a generalized Speh representation, then there exists a virtual finite dimensional representation $\sigma(\pi)$ of W_K with $\mathbb{Q}_+$ coefficients of dimension n s.t.
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma(\pi))\,\mathrm{tr}(h|\pi).$$
3. If $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$, then $\sigma(\pi)$ is a genuine representation of W_K.

Then (a) and (b) of(0.1.2.1) hold true for n, by defining $\sigma(\pi)$ as follows: If π has supercuspidal support $\{\pi_1,\ldots,\pi_r\}$(with multiplicity), then we define
$$\sigma_n(\pi) = \bigoplus \sigma(\pi_i)(\frac{1-n_i}{2}).$$

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Proof: □

Lemma (0.1.2.3) [Testing on Tempered Representations and Generalized Speh Representations].

- Let $d \in \mathbb{Z}_+, t \in \mathbb{Z}_{\ge 2}, n = dt, \pi_0 \in \mathrm{Irr}^{\mathrm{cusp,uni}}(\mathrm{GL}(d;K))$ and $\pi = \mathrm{St}(\pi_0,t)$. Then there exists $h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K))$ s.t. $\mathrm{tr}(h|\pi) = 0$ for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;F))$ that is tempered but not of the form $\pi = \pi_0(\mathrm{i}y_1) \times \ldots \times \pi_0(\mathrm{i}y_t)$ where $y_i \in \mathbb{R}$. And for these π, $\mathrm{tr}(h|\pi) \neq 0$.
- If $h \in C_c^\infty(\mathrm{GL}(n;K))$ s.t. for all $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(d;K))$ that is tempered but non-square-integrable or $\pi = \mathrm{St}(\pi_0,t)$ for some $d \in \mathbb{Z}_+, t \in \mathbb{Z}_{\ge 2}, n = dt, \pi_0 \in \mathrm{Irr}^{\mathrm{cusp,uni}}(\mathrm{GL}(d;K))$; then $\mathrm{tr}(h|\pi) = 0$ for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$.

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Proof: Kazhdan's density theorem**?** and □

Prop. (0.1.2.4)[Constructing Galois Representations]. The hypotheses of(0.1.2.2) are true for any $n \in \mathbb{Z}_+$. In particular, (a) and (b) of(0.1.2.1) hold true for any $n \in \mathbb{Z}_+$. ⌟

Proof: 1 is proved in Theorem 6.4. of [?], by relating the deformation spaces of one-dimensional p-divisible groups to the deformation spaces of their infinitesimal parts.

2. Firstly by(0.1.5.4), K can be realized as $K = F_w$ with notation as in(0.1.5.1) and(0.1.5.3). Then we can apply(0.1.7.5) to $\pi^\vee$ to get $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ s.t.

- $H_\xi^*(\pi_f) \neq 0$(0.1.5.6).
- $\pi_{p,0}$ is unramified.
- $\pi_w = \pi \otimes \chi$ for some unramified quasi-character χ of $K^\times$.

Then we apply(0.1.7.2) to get

$$\mathrm{tr}(f_{\tau,h}^\vee|\pi^\vee \otimes \chi) = \frac{1}{a(\pi_f)}\,\mathrm{tr}\left(\tau|[H_\xi[\pi_f]] \otimes \chi_{\pi_{p,0}}\right)\mathrm{tr}\left(h^\vee|\pi^\vee \otimes \chi\right).$$

Then we can take $\sigma(\pi) = \frac{1}{a(\pi_f)}[H_\xi[\pi_f]]^\vee \otimes \chi_{\pi_{p,0}}^{-1} \otimes \chi^{-1}$**?**, which has dimension n by(0.1.6.1).

3: It suffices to show that $\sigma(\pi) \in K_0(\mathrm{Rep}^{\mathrm{fd}}(W_K))$. For this, Cf.[?]P705**?**. □

Bijective Correspondence for Supercuspidal Representations

Thm. (0.1.2.5)[Supercuspidal Representations are Realized on Lubin-Tate Spaces]. Let $[R\psi_n]$ be the alternating sum of the global section of the vanishing cycles for the Lubin-Tate tower, then it's endowed with an admissible action of the subgroup $A_{K,n} \subset \mathrm{GL}(n;K) \times D_{K,1/n}^\times \times W_K$ consisting of elements (γ, δ, σ) s.t.

$$|\det\gamma|^{-1}.|\,\mathrm{Nmrd}(\delta)|.|\,\mathrm{Art}_K^{-1}\sigma| = 1$$

.

Let $\rho \in \mathrm{Irr}^{\mathrm{adm}}(D_{K,1/n}^*)$ s.t. $\pi = \text{J-L}(\rho) \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$. Then

$$[R\psi](\rho) = (-1)^{n-1}\pi^\vee|_{\mathrm{GL}(n;\mathcal{O}_K)} \otimes \sigma(\pi) \in K_0\left(\mathrm{Rep}^{(\mathrm{adm,cont})}(\mathrm{GL}(n;\mathcal{O}_K) \times W_K)\right)\ (0.1.2.2)\textbf{?}.$$

⌟

Proof: Cf.[?]P704.**?**

This follows from(0.1.4.7).**?** Cf.[?]P704. □

Thm. (0.1.2.6)[Characterizing the Supercuspidal Correspondence]. Suppose for each $K \in p$-`NField` and $n \in \mathbb{Z}_+$, there is a map

$$\mathrm{rec}_n' : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \to \mathrm{Rep}^n(W_K)$$

s.t.

- For a quasi-character χ of $K^\times$, $\mathrm{rec}_1'(\chi) = \chi \circ \mathrm{Art}_K^{-1}$**??**.
- If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ is a constituent of $\pi_1 \times \ldots \pi_t$, then $\sigma(\pi) = \sigma(\pi_1) \oplus \ldots \oplus \sigma(\pi_t)$.
- For a quasi-character χ of $K^\times$ and $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$,

$$\mathrm{rec}_n'(\pi(\chi)) = \mathrm{rec}_n'(\pi) \otimes \mathrm{rec}_1'(\chi).$$

- If K'/K is a cyclic Galois extension of prime degree and $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$ with $\Pi = \mathrm{BC}_F^{F'}(\pi) \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K'))$, then $\mathrm{rec}'(\Pi) = \mathrm{rec}'(\pi)|_{W_{K'}}$.
- If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ and $\mathrm{rec}'(\pi)$ is unramified, then π is Iwahori-spherical.

Then

$$\mathrm{rec}'_n : \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K)) \to \mathrm{Rep}^n(W_K)$$

is a bijection. ⌟

Proof: **?** □

Prop. (0.1.2.7). The correspondences defined in(0.1.2.1) satisfies the conditions in(0.1.2.6). In particular, it induces a bijection

$$\mathrm{rec}'_n : \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K)) \cong \mathrm{Rep}^n(W_K)$$

⌟

Proof: **?** □

3 p-divisible Groups and Vanishing Cycles

Def. (0.1.3.1) [p-divisible Groups]. Let $S \in \mathcal{S}\mathrm{ch}/\mathcal{O}_K$ s.t. p is locally nilpotently**?**, then a ϖ-divisible $\mathcal{O}_K$-module H over S is a p-divisible group $G \in p\mathrm{Div}(S)$ together with $\iota : \mathcal{O}_K \to \mathrm{End}_S(H)$ s.t. the two induced actions of $\mathcal{O}_K$ on $\mathrm{Lie}_S(G)$ are equal.

For any ϖ-divisible $\mathcal{O}_K$-module G/S, we have $[K:\mathbb{Q}_p] | \mathrm{ht}(G)$**?**. So we can define the **$\mathcal{O}_K$-height** $\mathrm{ht}_{\mathcal{O}_K}(G) = \mathrm{ht}(G)/[K:\mathbb{Q}_p]$. ⌟

Def. (0.1.3.2) [Drinfeld-level-m-Structures]. A **Drinfeld-level-m-structure** on a 1-dimensional ϖ-divisible $\mathcal{O}_K$-module H/S is a tuple of sections $(X_1, \ldots, X_n) \in H[\varpi^m]$ s.t.

$$\sum_{i_1,\ldots,i_n \in \mathcal{O}_K/(\varpi^m)} [i_1 X_1 + \ldots + i_n X_n] = [H[\varpi^m]]$$

as relative Cartier divisors over S. ⌟

<u>*D*-Groups</u>

Def. (0.1.3.3) [Local Simple Unitary Groups]. For $D'/\mathbb{Q}_p$ is a semisimple algebra with maximal order $\mathcal{O}_{D'}$, define $D = \mathrm{Mat}(n;K) \times D'$ with maximal order $\mathcal{O}_D = \mathrm{Mat}(n;\mathcal{O}_K) \times \mathcal{O}_{D'}$.

Define $\mathbf{G}_{D'} = \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p}(\mathcal{O}_D^*)$ and $\mathbf{G}_D = \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} \mathrm{GL}(n) \times \mathbf{G}_{D'}$. ⌟

Def. (0.1.3.4) [D-Groups]. Let $S \in \mathcal{S}\mathrm{ch}$ s.t. p is locally nilpotently, then a **D'-group** over S is an étale p-divisible group H over S together with an action

$$\iota : \mathcal{O}_{D'}^{\mathrm{op}} \to \mathrm{End}(H)$$

s.t. $H[p]$ is free of rank 1 over $\mathcal{O}_{D'}^{\mathrm{op}}/(p)$. ⌟

Def. (0.1.3.5) [$(\mathcal{O}_K, D')$-Groups]. For $S \in \mathcal{S}\mathrm{ch}/\mathcal{O}_K$, an $(\mathcal{O}_K, D')$-group over S is a p-divisible group $\underline{H}/S$ with an action

$$\iota : \mathcal{O}_D^{\mathrm{op}} \to \mathrm{End}(\underline{H})$$

s.t. $\underline{H}$ decomposes as a product $\underline{H} = H^{\oplus n} \times H'$ under the action of $\mathcal{O}_D^{\mathrm{op}} \cong \mathrm{Mat}(n; \mathcal{O}_K)^{\mathrm{op}} \times \mathcal{O}_{D'}^{\mathrm{op}}$ where H is a 1-dimensional ϖ-divisible $\mathcal{O}_K$-module of height n and H' is a D'-group(0.1.3.4).

And a **level-m-structure** on a $(\mathcal{O}_K, D')$-group $\underline{H}$ is a Drinfeld-level-m-structure on H and an isomorphism of $\mathcal{O}_{D'}^{\mathrm{op}}$-modules $\mathcal{O}_{D'}^{\mathrm{op}}/(p^m) \cong H'[p^m]$?. ⌟

Prop. (0.1.3.6) [Dieudonné Parameters]. An $(\mathcal{O}_K, D')$-group $\underline{H}$ over κ_r is equivalent to a Dieudonné module $\underline{M}$ over $W(\kappa_r)$ s.t. there is an isomorphism of $\mathcal{O}_D \otimes W(\kappa_r)$-moduele

$$\underline{M} \cong \mathcal{O}_D \otimes W(\kappa_r).$$

Thus $\underline{M}$ is represented by an element $\delta_0 \in \mathbf{G}_D(W(\kappa_r)[\frac{1}{p}])$ up to σ_K-conjugacy. This δ_0 is called the **Dieudonné paramter** of $\underline{H}$. ⌟

Deformations of p-adic Spaces

Def. (0.1.3.7) [$\overline{H}_\beta$]. Take $r \in \mathbb{Z}_+$ and a 1-dimensional ϖ-divisible $\mathcal{O}_K$-module $\overline{H}_\beta$ of height n over κ_r, whose Dieudonné module is given by an element

$$\beta \in \mathrm{GL}(n; \mathcal{O}_{K_r}) \,\mathrm{diag}(\varpi_K, 1, \dots, 1) \,\mathrm{GL}(n; \mathcal{O}_{K_r})$$

up to σ_K-conjugacy. ⌟

Def. (0.1.3.8) [Deformation Spaces]. Let $\mathrm{Spf}\, R_\beta$ be the formal deformation space of $\overline{H_\beta}$ as ϖ-divisible $\mathcal{O}_K$-modules over $\mathcal{O}_{K_r}$, with universal deformation H_β.?

For $m \in \mathbb{Z}_+$, let $R_{\beta,m}$ be the covering of R_β parametrizing Drinfeld-level-m-structures on H_β. ⌟

Prop. (0.1.3.9).

- $R_\beta \cong \mathcal{O}_{K_r}[[T_1, \dots, T_{n-1}]]$.
- $R_{\beta,m}/R_\beta$ is a finite Galois covering with Galois group $\mathrm{GL}(\mathcal{O}_K/(\varpi^m))$, étale in the generic fiber.
- $R_{\beta,m}$ is regular.

⌟

Proof: Cf.[**?**]P672. □

Def. (0.1.3.10) [Deformation Spaces]. For a $(\mathcal{O}_K, D')$-group $\underline{\overline{H}}$ over κ_r, there is a deformation space $\mathrm{Spf}\, R_{\underline{\overline{H}}}$ of deformations of $\underline{\overline{H}}$ over $\mathcal{O}_{K_r}$ with universal deformation $\underline{H}$, and a covering map $R_{\underline{\overline{H}},m}/R_{\underline{\overline{H}}}$ parametrizing level-m-structures on the universal deformation. ⌟

Prop. (0.1.3.11) [Comparison of Deformation Spaces]. If $\underline{\overline{H}} \cong \overline{H}_\beta^{\oplus n} \otimes \overline{H}'$, then

- The canonical map $\mathrm{Spf}\, R_{\underline{\overline{H}}} \to \mathrm{Spf}\, R_\beta$ is an isomorphism.
- The canonical map $\mathrm{Spf}\, R_{\underline{\overline{H}},m} \to \mathrm{Spf}\, R_{\beta,m} \times_{\mathrm{Spf}\, R_\beta} (H'[p^m])^*$ is an isomorphism, where $(H'[p^m])^*$ is the set of $\mathcal{O}_D^{\mathrm{op}}$-generators of $H'[p^m]$. In particular, $\mathrm{Spf}\, R_{\underline{\overline{H}},m}/\,\mathrm{Spf}\, R_{\underline{\overline{H}}}$ is a finite Galois cover with Galois group $\mathrm{GL}(n; \mathcal{O}_K/(\varpi^m)) \times (\mathcal{O}_{D'}/(p^m))^*$.

⌟

Proof: This follows from the fact there are no deformations of a D'-group, by rigidity of étale covers.? □

Def. (0.1.3.12) [Lubin-Tate Tower]. For β basic, the tower

$$(\operatorname{Spf} R_{n,m})_{m\in\mathbb{Z}_+} = \left(\operatorname{Spf} R_{\beta,m} \otimes_{\mathcal{O}_r} \breve{\mathcal{O}}_K\right)_{m\in\mathbb{Z}_+}$$

is called the **Lubin-Tate tower** of height n. This doesn't depend on β, because there exists a unique 1-dimensional formal $\mathcal{O}_K$-module of height n over $\overline{\kappa}$ and a unique Drinfeld-level-m-structure for any m. ⌟

Prop. (0.1.3.13). There is a compatible action of $A_{K,n} \subset \mathrm{GL}(n;K)\times D_{K,1/n}^\times \times W_K$ consisting of elements (γ,δ,σ) s.t.

$$|\det\gamma|^{-1}.|\operatorname{Nmrd}(\delta)|.|\operatorname{Art}_K^{-1}\sigma| = 1$$

on the Lubin-Tate tower, where

- $\mathrm{GL}(n;K)$ acts on the Drinfeld-level-structures,
- $D_{K,1/n}^\times$ acts on the trivialization of the special fiber.
- W_K acts as Frobenius.

⌟

Vanishing Cycles

Def. (0.1.3.14) [Vanishing Cycles]. For $\mathcal{X} \in \mathcal{S}\mathrm{ch}^{\mathrm{ft}}/\mathcal{O}_{\breve{K}}$ with generic fiber $X/\breve{K}$, the formalism of vanishing cycles is used to relate the cohomology of $\overline{X}$ and the cohomology of X_{K^s} together the action of I_K.

Consider the diagram $\overline{X} \xrightarrow{i} X \xleftarrow{\overline{j}} X_{K_s}$ and $\Lambda \in \mathcal{A}\mathrm{b}^{\mathrm{fin}}$, then the **sheaf of vanishing cycle** of $\mathcal{X}/\mathcal{O}_{\breve{K}}$ is defined to be

$$\Psi_{\mathcal{X}}^\bullet(\Lambda) = i^* R^\bullet \overline{j}_* \underline{\Lambda} \in \mathcal{S}\mathrm{h}(\overline{X}_{\text{ét}}),$$

which has an action of I_K. ⌟

Prop. (0.1.3.15) [Proper case]. If $\mathcal{X}/\mathcal{O}_{\breve{K}}$ is proper, then proper base change implies $i^* : \mathrm{H}^p_{\text{ét}}(\mathcal{X}, R^q\overline{j}_*\underline{\Lambda}) \cong \mathrm{H}^p_{\text{ét}}(\overline{X}, i^*R\overline{j}_*\underline{\Lambda})$. And the spectral sequence for $\overline{j}$ implies

$$\mathrm{H}^p_{\text{ét}}(\overline{X}, \Psi^n(\Lambda)) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_{K^s}, \Lambda).$$

which is I_K-invariant. If moreover, $\mathcal{X}/\mathcal{O}_K$ is smooth, then by**??**, this spectral sequence degenerate at page 2 and $\Psi^n(\Lambda) = \begin{cases} \underline{\Lambda} & , n=0 \\ 0 & , n\geq 2 \end{cases}$, inducing an isomorphism

$$\mathrm{H}^n_{\text{ét}}(\overline{X}, \Lambda) \Longrightarrow \mathrm{H}^n_{\text{ét}}(X_{K^s}, \Lambda).$$

where I_K acts trivially. ⌟

Def. (0.1.3.16) [Formal Vanishing Cycles]. The vanishing cycle is defined verbatim in the Berkovich formal schemes setting, and the resulting sheaves are called **formal vanishing cycles**. But Scholze didn't pass to the maximal unramified extension(the Lubin-Tate tower) as Harris-Taylor did, and we can get more informations. ⌟

Thm. (0.1.3.17) [Formal Vanishing Cycles on the Deformation Tower]. Consider the formal vanishing cycles on the tower $\{\mathrm{Spf}\, R_{\beta,m}\}_{m\in\mathbb{Z}_+}$:

$$\Psi^i_\beta = \varinjlim_{m\in\mathbb{Z}_+} H^0(\overline{\mathrm{Spf}\, R_{\beta,m}}, \Psi^i_{\mathrm{Spf}\, R_{\beta,m}}\overline{\mathbb{Q}_\ell}).$$

Then

$$H^0(\overline{\mathrm{Spf}\, R_{\beta,m}}, R^i\psi_{\mathrm{Spf}\, R_{\beta,m}}\overline{\mathbb{Q}_\ell}) \in \mathrm{Rep}^{\mathrm{fd}}\left(W_{K_r}\times \mathrm{GL}(n;\mathcal{O}_K/(\varpi^m))\right),$$

and it vanishes unless $i\in[0,n-1]$.

In particular $\Psi^i_\beta, [\Psi_\beta]\in \mathrm{Rep}^{(\mathrm{cont},\mathrm{adm})}(W_{K_r}\times \mathrm{GL}(n;\mathcal{O}_K))$. ⌟

Proof: Cf.[**?**]P673. □

Prop. (0.1.3.18) [Isomorphism of Formal Vanishing Cycles]. Define the formal vanishing cycles as

$$\Psi^i_{\underline{H}} = \varinjlim_{m\in\mathbb{Z}_+} H^0(R_{\underline{H}}, \Psi^i_{\mathrm{Spf}\, R_{\underline{H},m}}\overline{\mathbb{Q}_\ell}),$$

then they has an action of $W_{K_r}\times \mathbf{G}_D(\mathbb{Z}_p)$. And by(0.1.3.11), there is a $W_{K_r}\times \mathbf{G}_D(\mathbb{Z}_p)$-equivariant isomorphism

$$\Psi^i_{\underline{H}} \cong \Psi^i_\beta\otimes C^\infty_c(\mathcal{O}^*_{D'};\overline{\mathbb{Q}_\ell}).$$

⌟

Proof:

□

Def. (0.1.3.19) [Vanishing Cycles on the Lubin-Tate Tower]. Define the vanishing cycles as

$$\Psi_n = \varinjlim_{m\in\mathbb{Z}_+} H^0(\overline{\mathrm{Spf}\, R_{n,m}}, \Psi^i_{\mathrm{Spf}\, R_m}\overline{\mathbb{Q}_\ell}),$$

with an action of $A_{K,n}$ induced by(0.1.3.13). Then $\Psi_n\in \mathrm{Rep}^{(\mathrm{adm},\mathrm{alg},\mathrm{cont})}(\mathrm{GL}(n;\mathcal{O}_K)\times \mathcal{O}^*_{D_{K,1/n}}\times I_K)$, and it vanishes unless $i\in[0,n-1]$. ⌟

Proof: The smoothness of $\mathcal{O}^*_{D_{K,1/n}}$-action follows from comparison theorem Corollary 4.5 of [Vanishing cycles for formal schemes. 2, Berkovich], Cf. proof of [**?**], Lemma II.2.8. The rest follows from(0.1.3.17). □

4 Cyclic Base Change

Prop. (0.1.4.1). K_r/K be the unramified extension of degree r, then for any $x\in \mathrm{GL}(n;K_r)$, we can define $Nx = x.x^\sigma.\ldots.x^{\sigma^{l-1}}$.

Then N defines an injection from the σ-conjugacy classes of $\mathrm{GL}(n;K_r)$ to the conjugacy classes of $\mathrm{GL}(n;K)$. In fact, if $\gamma = Nx$, then $G_{x,\delta}$ is an inner form of G_γ. ⌟

Proof: Cf.[Arthur-Clozel]P3. □

Prop. (0.1.4.2) [Cyclic Base Change]. If $\gamma = Nx$, we can define orbital integrals and twisted orbital integrals:

$$\mathrm{TOrb}_\delta(\varphi) = \int_{G_{\delta,\sigma}\backslash \mathrm{GL}(n;K_r)} \varphi(g^{-1}\delta g^\sigma)\frac{\mathrm{d}g_r}{\mathrm{d}t},$$

$$\mathrm{Orb}_\gamma(f) = \int_{G_\gamma\backslash \mathrm{GL}(n;K)} \varphi(g^{-1}\delta g)\frac{\mathrm{d}g}{\mathrm{d}t}.$$

Then for any $\varphi \in C_c^\infty(\mathrm{GL}(n;K_r))$, there exists $f \in C_c^\infty(\mathrm{GL}(n;K))$ s.t. for any regular $\gamma \in \mathrm{GL}(n;F)$,

$$\mathrm{Orb}_\gamma(f) = \begin{cases} \mathrm{TOrb}_{\gamma,\sigma}(\varphi) & ,\gamma = N\delta, \delta \in \mathrm{GL}(n;K_r) \\ 0 & ,\text{otherwise} \end{cases}.$$

⌟

Proof: Cf.[Arthur-Clozel]P20. □

<u>**Test Functions**</u>

Def. (0.1.4.3) [Test Functions]. For any $\tau \in W_{K_r}$ and $h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K);\mathbb{Q})$, define

$$\psi_{\tau,h}(\beta) = \begin{cases} \mathrm{tr}(\tau \times h^\vee|[\Psi_\beta]) & ,\beta \in \mathrm{GL}(n;\mathcal{O}_{K_r})\,\mathrm{diag}(\varpi_K,1,\ldots,1)\,\mathrm{GL}(n;\mathcal{O}_{K_r}) \\ 0 & ,\text{otherwise} \end{cases}.$$

Then $\varphi_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K_r);\mathbb{Q})$, and is independent of ℓ. ⌟

Proof: Cf.[**?**]P674**?**. □

Def. (0.1.4.4) [Base Change Test Function, $f_{\tau,h}$]. Define $f_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K);\mathbb{Q})$ s.t. it has matching twisted orbital integral with $\varphi_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K_r);\mathbb{Q})$**?** w.r.t. the Haar measures that give the hyperspecial subgroups volume 1. ⌟

Def. (0.1.4.5). For any $\tau \in W_{K_r}$ and $h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K);\mathbb{Q}), h' \in C_c^\infty(\mathcal{O}_D^*;\mathbb{Q})$, define the function

$$\varphi_{\tau,h,h'}(\delta_0) = \mathrm{tr}\left(\tau \times h^\vee \times h' \mid [\Psi_{\underline{\overline{H}}}]\right) \text{(0.1.3.18)}.$$

if $\underline{\overline{H}}$ has Dieudonné paramter δ_0(0.1.3.6), and 0 if there is not such $\underline{\overline{H}}$. Then by(0.1.4.3), $\varphi_{\tau,h,h'} \in C_c^\infty(\mathbf{G}_D(W(\kappa_r)[\frac{1}{p}]))$ and is independent of ℓ. ⌟

Prop. (0.1.4.6) [Base Change Test Functions]. $\varphi_{\tau,h,h'} \in C_c^\infty(\mathbf{G}_D(\mathbb{Q}_{p^r});\mathbb{Q})$ corresponds to $f_{\tau,h}^\vee \times h' \in C_c^\infty(\mathbf{G}_D(\mathbb{Q}_p);\mathbb{Q})$. ⌟

Proof: Cf.[**?**]P688.**?** □

Prop. (0.1.4.7) [Relation to Lubin-Tate space]. Since β is basic, we can associate $d = N(\beta) \in D_{K,1/n}^*$ with the same characteristic polynomial. Then for any $\tau \in W_{K_r}$ and $h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K))$, we have

$$\varphi_{\tau,h}(\beta) = \mathrm{tr}\left(\tau \times d^{-1} \times h^\vee|[R\psi]\right).$$

⌟

Proof: Cf.[**?**]P704. □

5 Simple Shimura Varieties

Notation (0.1.5.1).

- Let $F_0 \in \texttt{NField}$ be totally real with $2|[F_0 : \mathbb{Q}]$.
- Let $\tau \in \Sigma_{F_0}^{\infty}$ and $x_0 \in \Sigma_{F_0}^{\text{fin}}$.
- Let $\mathcal{K} \subset \mathbb{C}$ be an imaginary quadratic field s.t. the rational prime below x_0 splits in $\mathcal{K}$.
- Let $F = F_0\mathcal{K}$ and $S_F(x_0) = \{x, x^{\mathsf{c}}\}$.
- Let $n \in \mathbb{Z}_+$.

⌟

Prop. (0.1.5.2) [Simple Unitary Groups]. There exists $D \in \mathrm{Az}_F$ of dimension n^2 with an involution $*$ of second kind, and a homomorphism $h_0 : \mathbb{C} \to \mathbb{D}_\tau$ s.t.

- $x \to h_0(\mathrm{i})^{-1}x^* h_0(\mathrm{i})$ is a positive involution on $\mathbb{D} \otimes \mathbb{R}$.
- $\mathbb{D}$ splits at all places $v \in \Sigma_F \setminus \{x, x^{\mathsf{c}}\}$.
- The algebraic group $\mathbf{G}_0 \subset \mathrm{Res}_{F/F_0}(\mathbb{D})$ consisting of x s.t. $xx^* = 1$ is quasi-split at all non-split places of F_0, unitary of signature $(1, n-1)$ at τ and unitary of signature $(0, n)$ at all other infinite places of F_0.

Define

$$\mathbf{G} \in \mathcal{R}\mathrm{ed}\mathcal{G}\mathrm{rp}\,/\mathbb{Q} : R \mapsto \{\, g \in (\mathbb{D} \otimes R)^* \mid gg^* \in R^* \,\}.$$

⌟

Proof: Take $\mathbb{D} = B^{\mathrm{op}}$ as in [**?**]Lemma 1.7.1. □

Notation (0.1.5.3).

- Let $w \in \Sigma_F^{\text{fin}}$ with $u = w \cap \mathcal{K}$ and $p = w \cap \mathbb{Q}$ s.t. p is split in $\mathcal{K}$ and $w \notin \{x, x^{\mathsf{c}}\}$(i.e. $\mathbb{D}$ is split at w).
- Let $K = F_w$ with residue field κ.
- Choose $\ell \in \texttt{Prime} \setminus \{p\}$ and an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$.

⌟

Prop. (0.1.5.4) [Globalization]. Any $K \in p\text{-}\texttt{NField}$ can be realized as $K = F_w$ for some F and w with the setting as in(0.1.5.1) and(0.1.5.3). ⌟

Proof:

□

Prop. (0.1.5.5) [Shimura Varieties]. Situation as in(0.1.5.2), we can regard h_0 as a map $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$. Then the datum $(\mathbf{G}, h^{-1})$ defines a tower of Shimura varieties

$$\varprojlim_{\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)} \mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G}, h^{-1})$$

with reflex field $E = \tau(F)$. ⌟

Proof: By [**?**] and[**?**]P691. □

Def. (0.1.5.6) [Automorphic Vector Bundles]. For any $\xi \in \mathrm{Irr}^{\mathrm{adm}}_{\mathbb{Q}_\ell}(\mathbf{G})$, we can get a lisse sheaf $\mathcal{F}_{\xi,\mathcal{K}} \subset \mathrm{Loc}^{\text{ét}}_{\mathbb{Q}_\ell}(\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G}))$ for any open compact $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$ that is small enough. The action of $\mathbf{G}(\mathbf{A}_f)$ on $\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G})$ extends to $\mathcal{F}_{\mathcal{K}}$, and we can consider the cohomologies

$$H^*_\xi = \varinjlim_{\mathcal{K}} \mathrm{H}^*_{\text{ét}}(\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G}), \mathcal{F}_{\xi,\mathcal{K}}) \in \mathrm{Rep}^{(\mathrm{cont},\mathrm{adm})}(\mathrm{Gal}_F \times \mathbf{G}(\mathbf{A}_f)).$$

⌟

Integral Models

Prop. (0.1.5.7) [Integral Models for G]. By our hypothesis(0.1.5.3), if we denote $D' = \prod_{w'|u, w' \neq w} \mathbb{D}_{w'}$ and $D = \prod_{w'|u} \mathbb{D}_{w'}$, then $\mathcal{O}_D = \mathrm{Mat}(n; \mathcal{O}_K) \times \mathcal{O}_{D'}$, putting us in the situation of(0.1.3.3). And we have:

$$\mathbf{G}_{\mathbb{Q}_p} = (\mathbf{G}_D)_{\mathbb{Q}_p} \times \mathbf{G}_m = \mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}(n) \times (\mathbf{G}_{D'})_{\mathbb{Q}_p} \times \mathbf{G}_m (0.1.3.3).$$

In particular, this gives an integral model of $\mathbf{G}_{\mathbb{Q}_p}$.

For any $m \in \mathbb{N}$, we can define congruence subgroups

$$\mathcal{K}_p^m = (1 = \varpi^m \mathrm{Mat}(n; \mathcal{O}_K)) \times (1 + p^m \mathcal{O}_{D'}) \times \mathbb{Z}_p^* \subset \mathbf{G}(\mathbb{Q}_p).$$

⌟

Prop. (0.1.5.8) [PEL-Type Integral Models]. Consider the functor on $\mathcal{N}\mathrm{Sch}\,/\mathcal{O}_K$ mapping $S \in \mathcal{N}\mathrm{Sch}\,/\mathcal{O}_K$ to isomorphism classes of typles $(A, \lambda, \iota, \overline{\eta}^p, \overline{\eta}_p)$ where

- $A \in \mathcal{A}\mathrm{b}\mathcal{V}\mathrm{ar}\,/S$ is a projective Abelian scheme of relative dimension $n^2[F_0 : \mathbb{Q}]$ up to prime-to-p isogeny.
- $\lambda : A \to A^\vee$ is a polarization of degree prime to p.
- ι is a $*_{D^{\mathrm{op}}}$ homomorphism $\mathcal{O}_{D^{\mathrm{op}}} \to \mathrm{End}_S(A)$ satisfying the determinant condition.
- $\overline{\eta}^p$ is a $\mathcal{K}^p$-level structure away from p.
- η_p a level-m-structure on $\underline{H}(A)$ where $\underline{H}(A) \oplus \underline{H}(A)^\vee \cong T_p(A)$ corresponding to $\mathcal{O}_{D^{\mathrm{op}}} \otimes \mathbb{Z}_p \cong \mathcal{O}_D^{\mathrm{op}} \times \mathcal{O}_D$.

and when $\mathcal{K}^p$ is small, this functor is represented by a projective scheme $\mathcal{S}\mathrm{h}_{\mathcal{K}^p,m}$, and the generic fiber of $\mathcal{S}\mathrm{h}_{\mathcal{K}^p,m}$ is a disjoint union of $\#Ш^1(\mathbb{Q}, \mathbf{G})$ copies of the canonical model $\mathcal{S}\mathrm{h}_{\mathcal{K}_p^m \mathcal{K}^p}(\mathbf{G}, \{h^{-1}\})$ localized at w. ⌟

Proof: Cf.[?]Section3.1 and 3.4. □

Prop. (0.1.5.9) [Group Actions]. For any $\xi \in \mathrm{Irr}^{\mathrm{adm}}_{\mathbb{Q}_\ell}(\mathbf{G})$, we can associate lisse $\overline{\mathbb{Q}_\ell}$-sheaves on $\mathcal{S}\mathrm{h}_{\mathcal{K}^p,m}$ compatible with the sheaf $\mathcal{F}_{\xi,\mathcal{K}_p^m\mathcal{K}^p}$ on the generic fiber.

And there is an obvious action of $\mathcal{O}_D^* \times \mathbf{G}(\mathbf{A}_f^p)$ on these integral models compatible with these lisse sheaves. ⌟

Proof:

□

6 Langlands-Kottwitz Method

Prop. (0.1.6.1). For any $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ and $\xi \in \mathrm{Irr}^{\mathrm{adm}}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G})$, consider

$$H^*_\xi[\pi_f] \in \mathrm{Rep}(\mathrm{Gal}_F)(0.1.5.6).$$

Then a theorem of Kottwitz implies that π_f appears in either odd dimension of even dimension**?**. Thus $\pm[H_\xi][\pi_f]$ is an genuine representation.

And there exists $a(\pi_f) \in \mathbb{N}$ s.t. $|\dim[H_\xi][\pi_f]| = a(\pi_f)n$ for any ξ.**?** ⌟

Def. (0.1.6.2). Define

$$f = h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p \in C_c^\infty(\mathbf{G}(\mathbf{A}_f))$$

where

$$h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K);\mathbb{Q}), \quad h' \in C_c^\infty(\mathcal{O}_{D'}^*;\mathbb{Q}), \quad f^p = \mathbf{1}_{K^p g^p g^p}, \quad g^p \in \mathbf{G}(\mathbf{A}_f^p).$$

Fix $m \in \mathbb{Z}_+$ s.t. $h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*}$ is bi-$\mathcal{K}_p^m$-invariant. ⌟

For $\tau \in W_K^+$ with $\deg(\tau) = r$, we want to evaluate $\mathrm{tr}(\tau \times f|[H_\xi])$ via Lefschetz trace formula.**?**

Lemma (0.1.6.3). There is a canonical isomorphism

$$\widehat{\mathcal{O}}_{\mathrm{Sh}_{K^p,0},x} \cong R_{\underline{\overline{H}}_y} \otimes \breve{\mathcal{O}}_K.$$

⌟

Proof: Cf.[**?**]P697**?**. □

Thm. (0.1.6.4).

$$\mathrm{tr}(\tau \times h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p|[H_\xi]) = \sum_{(\gamma_0:\gamma,\delta)} c(\gamma_0:\gamma,\delta)\,\mathrm{Orb}_\gamma(f^p)\,\mathrm{TOrb}_{\delta\sigma_0}(\varphi_{\tau,h,h'} \times \mathbf{1}_{p^{-1}W(\kappa_r)^*})\,\mathrm{tr}(\gamma_0|\xi).$$

where

- $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ is a semisimple element that becomes elliptic in $\mathbf{G}(\mathbb{R})$,
- $\gamma \in \mathbf{G}(\mathbf{A}_f^p)$ s.t. $\gamma \sim \gamma_0 \in \mathbf{G}(\overline{\mathbb{Q}_\ell})$ for any $\ell \in \mathtt{Prime} \setminus \{p\}$,
- $\delta \in \mathbf{G}(F)$ with conditions.

⌟

Proof: Cf.[**?**]P698. □

Cor. (0.1.6.5).

$$n\,\mathrm{tr}(\tau \times h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p|[H_\xi]) = \mathrm{tr}(f_{\tau,h}^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p|[H_\xi]).$$

⌟

Proof: Cf.[**?**]P698. □

7 Galois Extensions attached to Automorphic Forms

Notation (0.1.7.1). Use notations as in Simple Shimura Varieties. ⌟

Cor. (0.1.7.2). Notation as in Simple Shimura Varieties, assume that $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ s.t. $H^*_\xi(\pi_f) \neq 0$, and in the decomposition $\pi_p = \pi_w \otimes \pi_p^w \otimes \pi_{p,0}$ corresponding to

$$\mathbf{G}(\mathbb{Q}_p) = \mathrm{GL}(n; F) \times (D')^* \times \mathbb{Q}_p^\times \text{(0.1.5.7)},$$

assume $\pi_{p,0}$ is unramified and let $\chi_{\pi_{p,0}} = \left(\pi_{p,0} \circ \mathrm{Art}_{\mathbb{Q}_p}^{-1}\right)|_{W_K}$ be the quasi-character of W_K. Then for any $\tau \in W_K^+$ and $h \in C_c^\infty(\mathrm{GL}(n; \mathcal{O}_K))$, we have

$$\mathrm{tr}(f^\vee_{\tau,h}|\pi_w) = \frac{1}{a(\pi_f)} \mathrm{tr}\left(\tau|\, [H_\xi[\pi_f]] \otimes \chi_{\pi_{p,0}}\right) \mathrm{tr}(h^\vee|\pi_w).$$

⌟

Proof: Cf.[?]P699**?** □

Def. (0.1.7.3)[Potentially Abelian and Algebraic Representations]. For $F \in$ `NField` and $n \in \mathbb{Z}_+$, a $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ is called **potentially Abelian** if there exists $\Sigma \in \mathrm{Rep}^n(W_F)$ s.t. $\Sigma|_{W_{F_v}} = \mathrm{rec}_v(\Pi_v)$ for any $v \in \Sigma_F$.

$\Sigma \in \mathrm{Rep}^{\mathrm{fd}}(W_F)$ is called **algebraic Weil representation** if for any $\tau : F \to \mathbb{C}$, $\Sigma|_{W_\tau}$ is of the form $z \mapsto z^p \overline{z}^q$. ⌟

Def. (0.1.7.4) [*L*-Algebraic Representations]. Call $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ L-algebraic if $\Pi_\infty(\frac{n-1}{2})$ is regular algebraic(i.e. has the same infinitesimal character as an algebraic representation of $\mathrm{Res}_{F/Q}\mathrm{GL}(n)$). ⌟

Thm. (0.1.7.5) [Harris-Taylor Globalization]. If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n)/K)$ that is either essentially square-integrable or generalized Speh, then there exists $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ s.t.

- $H^*_\xi(\pi_f) \neq 0$(0.1.5.6).
- $\pi_{p,0}$ is unramified.
- π_w is an unramified twist of π.

⌟

Proof: Cf.[?]Corollary VI.2.5 and Lemma VI.2.11**?**. □

Cor. (0.1.7.6) [A variant]. If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n)/K)$ is essentially square-integrable, then ⌟

Thm. (0.1.7.7)[Jacquet-Langlands plus Base Change, Harris-Taylor]. Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^c$.
- Π_∞ is regular algebraic(i.e., it has the same infinitesimal character as an algebraic representation of $\mathrm{Res}_{F/Q}\mathrm{GL}(n)$).
- Π_x is square-integrable.

Then there exists

- Some $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$,
- Some $\xi \in \mathrm{Rep}^{\mathrm{fd}}(\mathbf{G})$ s.t. $H^*_\xi[\pi_f] \neq 0$.

- Some algebraic Hecke character ψ of $\mathcal{K}$ satisfying the following: For any $w \in \Sigma_F^{\text{fin}}$ with $u = w \cap \mathcal{K}$ and $p = w \cap \mathbb{Q}$ s.t. p is split in $\mathcal{K}$ and $w \notin \{x, x^{\mathsf{c}}\}$, we have

$$\pi_w \cong \Pi_w, \quad \pi_{p,0} \cong \psi_u.$$

⌟

Proof: Cf.[?].?
Use Theorem VI.1.1 to produce an automorphic representation of $\mathbb{D}^*$. It continues to have properties analogous to (i) and (ii). Then Lemma VI.2.10, Theorem VI.2.9 and the properties of the base-change map established in Theorem VI.2.1 finish the proof. □

Thm. (0.1.7.8)[Theorem 10.6 of Scholze, Deligne-Brylinski(86)/Carayol(86)/Harris-Taylor[?]]. Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π is regular algebraic.
- Π_x is square-integrable for some $x \in \Sigma_F^{\text{fin}}$ that is split over F_0.

Then there exists $a \in \mathbb{Z}_+$ and $R(\Pi) \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^{an}(\mathrm{Gal}_K)$ s.t. for any $v \in \Sigma_F^{\text{fin}} \setminus S(\ell)$, we have

$$R(\Pi)|_{W_{F_v}} = a\sigma(\Pi_v)$$

Moreover, for each $v \in \Sigma_F^{\text{fin}}$, the representation Π_y is tempered. ⌟

Proof: This is Theorem C of Harris-Taylor.?
Using p-adic Hodge theory, it would be no problem to show that one can choose $a = 1$, cf. proof of Proposition VII.1.8 in [?]. Also, this holds for any CM field F, cf. proof of Theorem VII.1.9 of [?].? □

Cor. (0.1.7.9). Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π is L-algebraic(0.1.7.4).
- Π_x is square-integrable for some $x \in \Sigma_F^{\text{fin}}$ that is split over F_0.

Then there exists $a \in \mathbb{Z}_+$ and $R(\Pi) \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^{an}(\mathrm{Gal}_K)$ s.t. for any $v \in \Sigma_F^{\text{fin}} \setminus S(\ell)$, we have

$$R(\Pi)|_{W_{F_v}} = a\,\mathrm{rec}'(\Pi_v)$$

⌟

Proof: It suffices to find a Hecke character χ of F s.t. $\chi^{-1} = \chi^{\mathsf{c}}$ and $\chi_\infty(\frac{n-1}{2})$, which can be done in the proof of Corollary VII.2.8 of [?]?. □

Thm. (0.1.7.10). Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t. $\Pi^\vee \cong \Pi^{\mathsf{c}}, \Pi_\infty$ is regular L-algebraic and Π_x is supercuspidal, then

- If there exists $\Sigma \in \mathrm{Rep}^{\mathrm{alg},n}(W_F)$ s.t. $\Sigma|_{W_{F_v}} = \mathrm{rec}(\Pi_v)$ for $v \in \Sigma_F^\infty$ and a.e. $v \in \Sigma_F^{\text{fin}}$, then it holds for any $v \in \Sigma_F$.
- If F'/F is a cyclic extension of prime degree s.t. there is only one place x' over x, and the cyclic base change Π' of Π to F' is potentially Abelian and supercuspidal at x', then Π is potentially Abelian.

⌟

Proof: Cf.[?]Thm13.5? □

Thm. (0.1.7.11)[Harris-Taylor]. Let $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ be essentially square-integrable, then there exists $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π_∞ is regular algebraic.
- Π_x is supercuspidal.
- Π_w is an unramified twist of π.

⌟

Proof: Cf.[?]Corollary VI.2.6. □

8 Non-Galois Automorphic Induction

Thm. (0.1.8.1) [Non-Galois Automorphic Induction].

- Let $F_0^3/F_0^2/F_0^1$ be totally real fields and $2|[F_0:\mathbb{Q}]$. Denote $n=[F_0^2/F_0^1]$.
- Suppose F_0^3/F_0^1 is a solvable Galois extension.
- Let $\mathcal{K}$ be an imaginary quadratic field and $F^i = F_0^i\mathcal{K}$.
- Let $x \in \Sigma_{F^1}^{\mathrm{fin}}$ that is split in F_0^1 and inert in F^3.
- Let χ be a Hecke character of F^2 s.t.
  - $\overline{\chi} = \chi \circ \mathsf{c}$.
  - For any $v \in \Sigma_{F^2}^{\infty}$, $\chi_v(z) = z^{p_v}\overline{z}^{-p_v}$, and $p_v \neq p_{v'}$ if $v \neq v'$.
  - The stablizer of $\chi_x \circ \mathrm{Nm}_{F^3/F^2}$ in $\mathrm{Gal}(F^3/F^1)$ is equal to $\mathrm{Gal}(F^3/F^2)$.

Then there exists a potentially Abelian $\rho \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F^1)$ that is associated to $\mathrm{Ind}_{F^1}^{F^2}\chi$. ⌟

Proof:

□