Integral canonical models

sign V
$(n, 2)$

moduli of
ab. var.

$$\begin{tikzcd}
 & \mathrm{Sh}(\mathrm{GSpin}(V), \Omega^{\pm}) \arrow[r] \arrow[d] & \mathrm{Sh}(\mathrm{GSp}(W), \mathcal{H}_g^{\pm}) \\
\mathcal{M}^{K3} \arrow[r] & \mathrm{Sh}(\mathrm{SO}(V), \Omega^{\pm}) &
\end{tikzcd}$$

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~1 hr

Last time

$G = SO(V)$ or $GSpin(V)$ or $GSp(W)$

$K \subseteq G(\mathbb{A}_f)$ $\quad \mathcal{D} = \Omega^{\pm}$ or $\mathcal{H}_g^{\pm}$

$$Sh_K(G, \mathcal{D}) \xrightarrow{} \operatorname{Spec} \mathbb{Q}$$

$$Sh_K(G,\mathcal{D}) = \underbrace{G(\mathbb{Q}) \backslash (\mathcal{D} \times G(\mathbb{A}_f)/K)}$$

$K = K_p K^p$ $\quad L_p \subseteq V_{\mathbb{Q}_p}$ or $W_{\mathbb{Q}_p}$ self-dual

$K_p = \mathrm{Stab}_{G(\mathbb{Q}_p)}(L_p)$ $\quad p \neq 2$

(for $G = GSpin(V)$, take $K_p = GSpin(L_p)(\mathbb{Z}_p)$)

<u>Def</u> For $G \longrightarrow \operatorname{Spec} \mathbb{Q}_p$ conn. reductive gp., open compact $K_p \subseteq G(\mathbb{Q}_p)$ is <u>hyperspecial</u> if $\exists\ \mathcal{G} \longrightarrow \operatorname{Spec} \mathbb{Z}_p$ reductive gp. (smooth, affine, reductive fibers) s.t. $K_p = \mathcal{G}(\mathbb{Z}_p)$.

<u>Ex</u> $\mathcal{G} = SO(L_p)$ etc.

Kisin "Integral canonical models of Shimura varieties."

Milne's idea:

$$\mathrm{Sh}_{K_p} := \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}$$

Transition maps finite étale

$\Rightarrow$ Sh_{K_p} is a scheme

$(\operatorname{Spec} \varinjlim A_i = \varprojlim \operatorname{Spec} A_i)$

regular and $\mathrm{Sh}_{K_p} \longrightarrow \operatorname{Spec} \mathbb{Q}$ formally smooth

meaning all local rings are Noetherian and regular; I don't claim Sh_{K_p} is locally Noetherian

For K^p small, $\mathrm{Sh}_{K_p} \circlearrowleft \underbrace{K^p}_{\text{pro-finite gp. scheme}}$

$$\mathrm{Sh}_{K_p K^p} = \mathrm{Sh}_{K_p} / K^p.$$

<u>Def</u> Given a scheme $X \longrightarrow \operatorname{Spec} \mathbb{Q}$ regular and formally smooth, an <u>integral canonical model</u> (ICM) is $\mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}$ w/ <u>extension property</u> regular, formally smooth, separated s.t.

$\forall\, S \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}$ reg. + formally smooth, any $S_{\mathbb{Q}} \longrightarrow X$ extends (uniquely) to $S \longrightarrow \mathcal{X}$

<u>Lemma</u> Let $A = \varinjlim A_i$ be a filtered colimit over a diagram of étale R-algebras. Then $R \to A$ is flat and formally étale. If R is regular, then A is regular.

as above, not requiring R or A Noetherian only their local rings (eg. $\underbrace{\mathbb{F}_2 \times \mathbb{F}_2 \times \cdots}_{\infty}$)

$\boxed{5}$

$\mathrm{Sh}_{K_p} \longrightarrow \mathrm{Sh}_{K_p K^p}$ is a K^p torsor (for K^p small)

$\boxed{6, 7}$

$$\begin{tikzcd} & \text{ICM} \\ Sh_{K_p'}(GSpin, \Omega^{\pm}) \arrow[r] \arrow[d] & \mathcal{S}h_{K_p'} \arrow[d, dashed, "\exists !"] \\ Sh_{K_p}(SO, \Omega^{\pm}) \arrow[r] & \mathcal{S}h_{K_p} \end{tikzcd}$$

$$K^p \times \mathcal{S}h_{K_p} \longrightarrow \mathcal{S}h_{K_p}$$

$$\mathcal{S}h_{K_pK^p} := \mathcal{S}h_{K_p}/K^p$$

(K^p small)

<u>Thm</u> (Kisin '10, Vasiu, Kim-Madapusi '16)

ICM's exist for abelian type Shimura data at hyperspecial level (reflex field might not be $\mathbb{Q}$)

<u>Ex</u> $(G, \mathcal{D}) = (SO(V), \Omega^{\pm})$

$(GSpin(V), \Omega^{\pm})$

$(GSp(W), \mathcal{H}_g^{\pm})$

K_p as above

<u>Ex</u> (Hodge type) $(p \neq 2)$

$$(G, \mathfrak{D}) \hookrightarrow (GSp(W), \mathcal{H}_g^{\pm})$$

$K_p \subseteq G(\mathbb{Q}_p)$ hyperspecial

<u>Assume</u> $K_p \subseteq \mathrm{Stab}(L'_p)$

$$\mathrm{Stab}(L'_p) = K'_p \subseteq GSp(\mathbb{Q}_p)$$

$L'_p \subseteq W$ self-dual

(e.g. $G = GSpin(V)$ from before
is Hodge type;
assumption above holds at (at least)
all but finitely many p for
Kuga-Satake $(GSpin(V), \Omega^{\pm}) \hookrightarrow (GSp(W), \mathcal{H}_g^{\pm})$)

$$\begin{tikzcd} Sh_{K_pK^p}(G, \mathfrak{D}) \arrow[r, "\text{closure}", "\text{+ normalize}"'] \arrow[d] \arrow[dr] & \mathcal{S}h_K \arrow[d] \\ Sh_{K'_pK'^p}(GSp, \mathcal{H}_g^{\pm}) \arrow[r] & \mathcal{A}_{g,K'} \end{tikzcd}$$

Y. Xu '20 $\Rightarrow$ normalization redundant

$$\mathcal{S}h_{K_p} := \varprojlim_{K^p} \mathcal{S}h_{K_pK^p}$$

(See page 34 from last time.)