Integral models

§1

Def A K3 surface X over a field k is a smooth proper geom. conn. dim. 2 scheme $X \longrightarrow \operatorname{Spec} k$, s.t.

$$\omega_X \cong \mathcal{O}_X \quad H^1(X, \mathcal{O}_X) = 0.$$

($\Rightarrow$ projective)

If $k = \bar{k}$, a (primitive) polarization is an ample line bundle $\mathcal{L}$ s.t. $\mathcal{L} \neq \mathcal{E}^{\otimes n}$ $\forall n > 1$

NB $\emptyset$ isn't connected by convention [Stacks project, 0B4S]

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Learning seminar at MIT
Oct. 30, 2023

Thm (Tate conj. for K3's)

If k is f.g. over $\mathbb{Q}$ or $\mathbb{F}_p$, X is K3

$$CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2i}_{et}(X_{k^s}, \mathbb{Q}_\ell(i))^{Gal(k^s/k)}$$

for all i. $(\ell \neq p)$

(Madapusi '15, Kim-Madapusi '16, Maulik, Deligne, Charles, André...)

$$CH^1(X) \cong \operatorname{Pic}(X) \hookrightarrow \operatorname{Pic}(X_{k^s})$$ Nygaard-Ogus

Kummer:

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^{\ell^n}} \mathbb{G}_m \longrightarrow 1$$

$$\begin{tikzcd} H^1_{et}(X_{k^s}, \mathbb{G}_m) \arrow[r, "x \mapsto x^{\ell^n}"] & H^1_{et}(X_{k^s}, \mathbb{G}_m) \arrow[r, "cl"] \arrow[d, phantom, "\shortparallel"] & H^2_{et}(X_{k^s}, \mu_{\ell^n}) \\ & \operatorname{Pic}(X_{k^s}) & \end{tikzcd}$$

$$\operatorname{Pic}(X_{k^s}) \otimes \mathbb{Z}_\ell \hookrightarrow H^2_{et}(X_{k^s}, \mathbb{Z}_\ell(1))$$

X is K3 $\Rightarrow$ $\operatorname{Pic}(X_{k^s})$ is ℓ-torsion free

(Deligne)

Lift to char. 0 (say $k = \bar{k}$ in char. $p > 0$)

$\Rightarrow H^i_{et}(X_{k^s}, \mathbb{Z}_\ell)$ free over $\mathbb{Z}_\ell$

rank	1,	0,	22,	0,	1
i	0,	1,	2,	3,	4

See Rizov "Moduli stacks of polarized K3 surfaces..."

Lemma If X is K3 over any k,

$\exists k'/k$ fin. sep s.t. $\forall K/k'$,

$$\mathrm{Pic}(X_{k'})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}(X_K)_{\mathbb{Q}}$$

Pf $\mathrm{Pic}^{\circ}(X_{\bar{k}}) = 1$ $\mathrm{Pic}(X_{\bar{k}}) = NS(X_{\bar{k}})$ is f.g.

$\underline{\mathrm{Pic}}(X)$ is "discrete", locally fin. type

⇒ assume K/k finite

$$\mathrm{Pic}(X_{k'})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}(X_K)_{\mathbb{Q}}$$

if K/k' purely insep.

($X_{k'} = X_K$ as top space

$\{f_{ij}\}$ cocycle over K for $\mathcal{L}$

$\Rightarrow \{f_{ij}^{p^n}\}$ cocycle over k' for $\mathcal{L}^{\otimes p^n}$, $n \gg 0$

Fact For k'/k finite Galois,

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{Pic}(X_{k'})_{\mathbb{Q}}^{\mathrm{Gal}(k'/k)}$$

(Hochschild-Serre)

$$\begin{tikzcd}
\mathrm{Pic}(X)_{\mathbb{Q}_\ell} \arrow[r] \arrow[d, equal] & H^2_{et}(X, \mathbb{Q}_\ell(1))^{G_k} \arrow[d, equal] \\
(\mathrm{Pic}(X_{k'})_{\mathbb{Q}_\ell})^{\mathrm{Gal}(k'/k)} \arrow[r] & (H^2_{et}(X, \mathbb{Q}_\ell(1))^{G_{k'}})^{\mathrm{Gal}(k'/k)}
\end{tikzcd}$$

(surfaces, $NS(X)$ ...)

Lemma If X is K3 over any k,

$$\mathrm{Pic}(X) \xrightarrow{\sim} NS(X) \xrightarrow{\sim} \mathrm{Num}(X)$$

($\Rightarrow \mathrm{Pic}^{\circ}(X_{\bar{k}}) = 1$)

Pf WTS $(\mathcal{L}, \mathcal{E}) = 0 \ \forall \mathcal{E} \Rightarrow \mathcal{L} = \mathcal{O}_X$

If $\mathcal{E}$ ample, $H^0(X, \mathcal{L}) = 0$.

$\Rightarrow H^2(X, \mathcal{L}) = 0$.

$0 \geq \chi(X, \mathcal{L}) := \frac{1}{2}(\mathcal{L}, \mathcal{L}) + 2$

$\Rightarrow (\mathcal{L}, \mathcal{L}) < 0$.

See Huybrechts "Lectures on K3 surfaces"

§2 Given $V = \bigoplus V^{p,q}$

$$V(1) := \bigoplus V(1)^{p,q}$$

$$V(1)^{p,q} = V^{p+1,q+1}$$

$$\mathrm{Hom}(V_1, V_2) = \bigoplus \mathrm{Hom}(V_1, V_2)^{p,q}$$

$$\mathrm{Hom}(V_1,V_2)^{p,q} := \bigoplus_{\substack{p_2 - p_1 = p \\ q_2 - q_1 = q}} \mathrm{Hom}(V_1^{p_1,q_1}, V_2^{p_2,q_2})$$

$\mathbb{C}^\times \curvearrowright V_i$, $\quad \underset{a}{\mathbb{C}^\times} \curvearrowright \mathrm{Hom}(V_1, V_2)$

$$f \longmapsto a \circ f \circ a^{-1}$$

<u>Ex</u> $(H^2(X_{\mathbb{C}}, \mathbb{C})(1))^{0,0}$

$$\cong H^{1,1}(X_{\mathbb{C}}, \mathbb{C})$$

$\mathrm{End}(V)^{0,0}$

$$\cong \{ f \in \mathrm{End}(V) : f(V^{p,q}) \subseteq V^{p,q} \}.$$

Sketch (num. field)

K3 $X \rightsquigarrow A$ Kuga-Satake ab. var.

$$\begin{tikzcd}
\mathrm{Pic}(X_{k'})_{\mathbb{Q}} \arrow[r] & H^2(X_{\mathbb{C}},\mathbb{C})(1) \arrow[r, hook, "\text{Hodge}"] & \mathrm{End}(H^1(A_{\mathbb{C}},\mathbb{C})) \arrow[l, bend right, "\pi"'] & \\
\mathrm{Pic}(X)_{\mathbb{Q}} \arrow[u] \arrow[r] \arrow[d] & H^2(X_{\mathbb{C}},\mathbb{Q})(1) \arrow[u] \arrow[d] \arrow[r, hook] & \mathrm{End}(H^1(A_{\mathbb{C}},\mathbb{Q})) \arrow[u] \arrow[d] \arrow[l, bend right] & \\
\mathrm{Pic}(X)_{\mathbb{Q}_\ell} \arrow[r] \arrow[dr, hook, "\sim"] & H^2_{et}(X_{k^s},\mathbb{Q}_\ell(1)) \arrow[r, hook, "\text{Galois}", "\text{Tate}"'] & \mathrm{End}(H^1_{et}(A_{k^s},\mathbb{Q}_\ell)) \arrow[l, bend right] & \\
 & H^2_{et}(X_{k^s},\mathbb{Q}_\ell(1))^{G_k} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, hook] & \left(\;''\quad''\quad''\;\right)^{G_k} \arrow[u, phantom, sloped, "\subseteq"] \overset{\text{Faltings}}{\cong} \mathrm{End}(A)\otimes\mathbb{Q}_\ell & \\
 & \alpha \arrow[r, mapsto] & \sum b_i\cdot f_i,\ f_i\in\mathrm{End}(A) &
\end{tikzcd}$$

See Huybrechts §17

$f_i \in \mathrm{End}(H^1(A_{\mathbb{C}}, \mathbb{Q}))^{0,0}$

$\pi(f_i) \in H^{1,1}(X_{\mathbb{C}}, \mathbb{C})$

Lefschetz (1, 1) $\Rightarrow \pi(f_i) \in \mathrm{Pic}(X^{an}_{\mathbb{C}})$

$\qquad \mathrm{Pic}(X_{\mathbb{C}})$ ($\shortparallel$ above)

Lemma $\Rightarrow \pi(f_i) \in \mathrm{Pic}(X_{k'})$

k'/k sep.

$\sum b_i \pi(f_i) \in \mathrm{Pic}(X_{k'})^{Gal(k'/k)}_{\mathbb{Q}_\ell}$

$\mapsto \alpha \qquad \overset{\shortparallel}{\mathrm{Pic}}(X)_{\mathbb{Q}_\ell}$

$V = L_d \otimes \mathbb{Q}$

A "=" $Cl(V_{\mathbb{R}})/Cl(L_d)$

$V \hookrightarrow Cl(V) = (\bigoplus T^{\otimes n})/(v \otimes v - (v,v))$.

$\mathrm{GSpin}(V) \subseteq Cl(V) \circlearrowleft$

$a \qquad x \mapsto a \cdot x$

$$\begin{tikzcd}
 & Sh(\mathrm{GSpin}(V), \Omega) \arrow[r] \arrow[d] & Sh(\mathrm{GSp}(W), \mathcal{H}_g^{\pm}) \\
\mathcal{M}^{K3} \arrow[r] & Sh(SO(V), \Omega) &
\end{tikzcd}$$

$W = Cl(V)$

$2g = \dim W = 2^{\dim V}$

$(X, \xi)/\mathbb{C}$

$V = \langle \xi \rangle^{\perp} \subseteq H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

$= PH^2(X, \mathbb{Q}) \quad$ sig. $(2, 19)$

$X \overset{?}{\rightsquigarrow} A$ over any k

$H^1_{et}(A_{k^s}, \mathbb{Z}_\ell) \overset{?}{\cong} Cl(PH^2_{et}(X_{k^s}, \mathbb{Z}_\ell(1)))$

$PH^2_{et}(X_{k^s}, \mathbb{Z}_\ell(1)) \overset{?}{\hookrightarrow} End(H^1_{et}(A_{k^s}, \mathbb{Z}_\ell))$

⌜Imagine: $k = \mathbb{F}_p$

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{X} \arrow[d, "\pi"] & X \arrow[l, "\text{lift}"'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\operatorname{Spec} \mathbb{Q} \arrow[r] & \operatorname{Spec} \mathbb{Z}_{(p)} & \operatorname{Spec} \mathbb{F}_p \arrow[l]
\end{tikzcd}$$

Deligne: over k perfect, can lift K3 to some W' finite flat over $W(k)$

Alternately: flatness of some moduli spaces...

$$\begin{tikzcd}
\tilde{A}^{ks} \arrow[r, "\text{extend}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{A} \arrow[d, "\pi_{ks}"] & A \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\operatorname{Spec} \mathbb{Q} \arrow[r] & \operatorname{Spec} \mathbb{Z}_{(p)} & \operatorname{Spec} \mathbb{F}_p \arrow[l]
\end{tikzcd}$$

$PR^2_{et}\pi_* \mathbb{Z}_\ell(1) \xrightarrow{?} End(R^1_{et}\pi_{ks,*}\mathbb{Z}_\ell)$

over $\mathbb{Z}_{(p)}$? ⌟

Equiv. to do over $\mathbb{Q}$:

finite loc. constant sheaves

↕

finite étale schemes

Lemma $\mathrm{F\acute{E}t}/\mathbb{Z}_{(p)} \longrightarrow \mathrm{F\acute{E}t}/\mathbb{Q}$

$$Y \longmapsto Y_{\mathbb{Q}}$$

is fully faithful

Pf $\left\{\begin{array}{l}\text{finite}\\ \pi_1(\mathbb{Z}_{(p)}, \overline{\mathbb{Q}})\text{-sets}\end{array}\right\} \to \left\{\begin{array}{l}\quad ''\\ \pi_1(\mathbb{Q}, \overline{\mathbb{Q}})\text{-}''\end{array}\right\}$

fully faithful b/c

$$\begin{tikzcd} \pi_1(\mathbb{Q}, \overline{\mathbb{Q}}) \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & \pi_1(\mathbb{Z}_{(p)}, \overline{\mathbb{Q}}) \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, two heads] & \mathrm{Gal}(\mathbb{Q}^{unr,p}/\mathbb{Q}). \end{tikzcd}$$

Rmk True for any Noeth. normal integral S, $\mathrm{Spec}\, K(S) \longrightarrow S$.

[Stacks Project, 0BQM]

§3 **Def** A (primitively) polarized K3 **K3 scheme** (X, ξ) over S is

$X \xrightarrow{\pi} S$ proper smooth

$\xi \in \mathrm{Pic}_{X/S,et}(S)$ s.t. (X, ξ) is (prim.) polarized K3 surface in geom. fibers

Def $\deg(\xi) := (\xi, \xi) = 2d$, $d \in \mathbb{Z}$ (locally constant)

MathOverflow question 208839...

(Alg. spaces...)

Def Moduli stack (prim. polar.)

$$\mathcal{M}^{K3}_{2d}(S) = \left\{ (X, \xi) \ \underset{\deg \xi = 2d}{\text{K3}} \text{ over } S \right\}$$

Fact $\mathcal{M}^{K3} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is separated, Deligne-Mumford, finite type, smooth over $\operatorname{Spec} \mathbb{Z}[1/(2d)]$

$\Bigg\{$ If $\mathcal{L} \in \operatorname{Pic}(X)$ is ample
$\Rightarrow \mathcal{L}^{\otimes 3}$ rel. very ample
$\pi_* \mathcal{L}^{\otimes 3}$ loc. free rank $9d+2$

See Huybrechts §5

Rmk "K3 surface" is open property:
Given any proper flat $X \longrightarrow S$ of finite presentation, if X_s is K3 for some $s \in S$
$\Rightarrow X_U \to U$ is K3 for some nbhd U

Pf Smooth locus is open on X, dim. loc. const.

$$\left.\begin{array}{l} h^0(X_{s'}, \mathcal{O}_{X_{s'}}) = 1 \\ h^1(X_{s'}, \mathcal{O}_{X_{s'}}) = 0 \end{array}\right\} \text{nearby} \quad \text{upper semi-cont.}$$

$$h^0(X_{s'}, \omega_{X_{s'}}) = h^0(X_{s'}, \omega_{X_{s'}}^{-1}) = h^2(X_{s'}, \mathcal{O}_{X_{s'}}) = h^2(X_{s'}, \omega_{X_{s'}}^{\otimes 2}) = 1$$

nearby (cohom. and base change)

$\Rightarrow H' \subseteq \underline{\mathrm{Hilb}}_{\mathbb{P}^N}$ open locus of K3 surfaces, $H \subseteq H'$ open "$\mathcal{O}(1)|_X \cong \mathcal{L}^{\otimes 3}$"

$$H(S) = \{ (Z \overset{\iota}{\hookrightarrow} \mathbb{P}^N_S) \in H'(S) :$$

1) $\iota^*\mathcal{O}(1) \cong \mathcal{L}^{\otimes 3} \in \mathrm{Pic}(Z)/\pi^*\mathrm{Pic}(S)$
2) $\mathcal{L}$ is prim. in geom. fibers
3) $H^0(\mathbb{P}^N_{k(s)}, \mathcal{O}(1)) \xrightarrow{\sim} H^0(Z_s, \mathcal{L}_s^{\otimes 3}) \}$

$$[H/PGL_N] \xrightarrow{\sim} \mathcal{M}^{K3}_{2d}$$

$N = 9d + 3$

See Huybrechts

§4 **Def** Moduli stack

$$\mathcal{A}_g(S) = \{ (A, \lambda) : \begin{array}{l} A \to S \text{ dim. } g \text{ ab. sch.} \\ \lambda : A \to A^\vee \text{ princ. pol.} \end{array} \}$$

Fact $\mathcal{A}_g \longrightarrow \mathrm{Spec}\,\mathbb{Z}$

is separated, Deligne-Mumford, finite type, smooth

Ex Level N over $\mathrm{Spec}\,\mathbb{Z}[1/N]$

$$A[N] \xleftarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$$

symplectic up to scalar.

Λ rank $2g$ $\mathbb{Z}$-lattice,
self-dual symplectic $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$GSp := GSp(\Lambda) \longrightarrow \operatorname{Spec} \mathbb{Z}$

$K(N) := \ker(GSp(\hat{\mathbb{Z}}) \to GSp(\mathbb{Z}/N\mathbb{Z}))$

For $S/\operatorname{Spec}[1/N]$, conn, $\bar{s} \longrightarrow S$ char. p

Def Level K on $A \longrightarrow S$:

$\left\{\begin{array}{l} K\text{-orbit of } \eta : T^p(A_{\bar{s}}) \xleftarrow{\sim} \Lambda \otimes \hat{\mathbb{Z}}^p \\ \text{sympl up to scalar} \end{array}\right.$

$\pi_1(S, \bar{s})$-stable

(say $K = \prod K_\ell$
For $K_\ell \neq GSp(\mathbb{Z}_\ell)$,
$S \longrightarrow \operatorname{Spec}[1/\ell]$)

$\eta \otimes \mathbb{Z}/N\mathbb{Z} \iff A[N] \xleftarrow{\sim} \Lambda \otimes \mathbb{Z}/N\mathbb{Z}$

$A[N] \longleftrightarrow T^p(A_{\bar{s}}) \otimes \mathbb{Z}/N\mathbb{Z}$

$F\acute{E}t/S \longleftrightarrow \left\{ \begin{array}{l} \text{finite} \\ \pi_1(S,\bar{s}) - \text{sets} \end{array} \right\}$

Def $\Big\{$ For $K \subseteq K(1)$ open compact,

$\mathcal{A}_{g,K}(S) := \left\{ (A, \lambda, \eta) : \begin{array}{l} (A,\lambda) \in \mathcal{A}_g(S) \\ \eta \text{ level } K \text{ structure} \end{array} \right\}$

Fact: If $K \subseteq K(N)$ $N \geq 3$,
$\mathcal{A}_{g,K} \longrightarrow \operatorname{Spec} \mathbb{Z}[1/N]$
is smooth, quasi-proj.

Can be deduced from
Mumford GIT book, Thm 7.9
cf. MathOverflow Question 6482

For $K \subseteq K(N)$, $N \geq 3$

$$\begin{tikzcd}
\overset{(A,\lambda,\eta,\varphi)}{\mathcal{A}_{g,K,\mathbb{C},\text{framed}}} \arrow[r, "\sim"] \arrow[d] & \mathcal{H}_g^{\pm} \times G(\mathbb{A}_f)/K \arrow[d] \\
\mathcal{A}^{an}_{g,K,\mathbb{C}} \arrow[r, "\sim"] & G(\mathbb{Q})\backslash(\mathcal{H}_g^{\pm} \times G(\mathbb{A}_f)/K) = Sh_K(GSp, \mathcal{H}_g^{\pm}) \\
(A,\lambda,\eta) \arrow[r, mapsto] & (J, (\varphi \otimes \mathbb{A}_f) \circ \eta) \\
\varphi : H_1(A,\mathbb{Q}) \xrightarrow{\sim} \Lambda \otimes \mathbb{Q} & \\
\mathcal{H}_g^{\pm} = \left\{ Z \in Sym_{g \times g}(\mathbb{C}) : \begin{matrix} im(Z) > 0 \\ \text{or } im(Z) < 0 \end{matrix} \right\} \arrow[uuuu, bend left] &
\end{tikzcd}$$

sympl up o scalar

holomorphic bijection

($\Rightarrow$ holo. iso.)

$G = GSp$

Level for $\mathcal{M}^{K3}_{2d}$:

$L := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$

$\rotatebox[origin=c]{90}{$\subseteq$}$

$L_d = \langle e_d \rangle^{\perp}, \quad e_d \in L$

$(e_d, e_d) = 2d$

For $K \subseteq SO(L_d)(\mathbb{A}_f)$

open compact open w/ admissible

$L_{\hat{\mathbb{Z}}}$ K-stable $(SO(L_d) \hookrightarrow SO(L))$

$S / \operatorname{Spec} \mathbb{Z}_{(p)}$ conn., $\bar{s} \to S$ char. $p \neq 2$

Def Level K on $(X, \xi) \in \mathcal{M}^{K3}_{2d}$:

$\{$K-orbit of $\eta : L \otimes \hat{\mathbb{Z}}^p \xrightarrow{\sim} H^2_{et}(X_{\bar{s}}, \hat{\mathbb{Z}}^p(1))$
isometry w/ $\eta(e_d) = c_1(\xi)$

$\pi_1(S, \bar{s})$-stable

(Some $S \longrightarrow \operatorname{Spec}[1/\ell]$ condition like before)

Def Moduli stack $\mathcal{M}^{K3}_{2d,K}$ w/ level K (admissible)

Fact For $p \nmid 2d$, $K_p = SO(L_d)(\mathbb{Z}_p)$
$K^p \subseteq SO(L_d)(\mathbb{A}_f)$

$K = K_p K^p$ admissible, neat

$\mathcal{M}^{K3}_{2d,K} \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}$

smooth scheme
separated finite type

$\mathcal{M}^{K3}_{2d,K} \longrightarrow \mathcal{M}^{K3}_{2d}$ finite étale

(after inverting ℓ where K^p isn't "standard")

Thm (Kisin '10, Vasiu) $p \nmid 2d$

For $G = GSpin(L_d)$
or $SO(L_d)$

$K_p = G(\mathbb{Z}_p)$

$K^p \subseteq G(\mathbb{A}_f^p)$ small

$Sh_K(G, \Omega)$ has a
smooth integral canonical model
$\mathcal{S}_K(G, \Omega)$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$

Kisin09 - "Integral canonical models of Shimura varieties"
Kisin10 - "Integral models ... abelian type"

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cf. [Mad16, Lemma 2.6] also
and intro to [Mad15], cf. Tom Lovering
Mad16 = Madapusi - "Integral canonical models ..."
Mad15 = " " - "Tate conjecture..."

(cont...) When $G = GSpin$,
$\mathscr{S}_K(G,\Omega)$ is normalization
of closure $Sh_K(G,\Omega) \hookrightarrow Sh_{K'}(GSp, \mathcal{H}_g^{\pm})$
for some K'

Xu '20 ⇒ normalization redundant
(Yujie)

Milne's idea: require

$$\mathscr{S}_{K_p} := \varprojlim_{K^p} \mathscr{S}_{K_pK^p}(G,\Omega)$$

to have univ. property:
for any $S \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}$
regular, formally smooth,

$$S \otimes \mathbb{Q} \longrightarrow \mathscr{S}_{K_p} \otimes \mathbb{Q} = Sh_{K^p}$$

extends to

$$S \longrightarrow \mathscr{S}_{K_p}$$

(supposed to characterize $\mathscr{S}_{K_p}$)

(extension is unique
at least w/ some separated
hypotheses)

This is the meaning of "integral
canonical model"

Then "take quotient of tower"
to recover finite level...

reduced + formally smooth over $\mathbb{Z}_{(p)}$
⇒ scheme-theoretic closure of generic fiber
⇔ flat

(e.g. GSpin), transition maps in $\varprojlim_{K^p} \mathcal{S}_{K_p K^p}$ are finite étale

$\Rightarrow \mathcal{S}_{K_p}$ is formally smooth scheme w/ all local rings regular

("global" Noetherianity ...?)

can have non-Noetherian rings w/ all local rings regular Noetherian, e.g.

$$\underbrace{\mathbb{F}_\ell \times \mathbb{F}_\ell \times \cdots}_{\infty}$$

Milne "Points on a Shimura variety modulo a prime of good reduction"

Ex $K_p = GSp(\mathbb{Z}_p)$

$W = \mathbb{Z}_p^{unr}$ $F = \mathbb{Q}_p^{unr}$

$$\left(\varprojlim_{K^p} \mathcal{A}_{g, K_p K^p}\right)(F) = \left(\varprojlim \text{ '' }\right)(W)$$

$$\overset{\shortparallel}{\left\{ (A, \lambda, \eta) : \eta : T^p(A_{\bar{F}}) \xrightarrow[\sim]{\text{sympl. up to scalar}} \Lambda \otimes \hat{\mathbb{Z}}^p \right\}}$$

$\pi_1(F, \bar{F})$-stable

$\shortparallel$

$\mathrm{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p^{unr})$

Descend to $(\tilde{A}, \tilde{\lambda})$ over $E/\mathbb{Q}_p$ finite unr.

$\mathrm{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p^{unr})$ acts trivially on $T_\ell(\tilde{A})$

Néron - Ogg - Shafarevich

$\Rightarrow \tilde{A}$ has good reduction

so (A, λ, η) extends to W.

To be updated; I would like to change $\mathbb{Z}_p$ to $\mathbb{Z}_{(p)}$

(Is $\mathrm{Spec}\, \mathbb{Z}_p \longrightarrow \mathrm{Spec}\, \mathbb{Z}_{(p)}$ formally smooth?)