GGP Conjecutures for Unitary Groups:Bessel and Fourier-Jacobi Periods

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Contents

1 Review of Rankin-Selberg method

Let F be a number field, $\mathbb{A} = \mathbb{A}_F$ be the adèle ring. $\pi = \pi_1 \boxtimes \pi_2$ be a cuspidal automorphic representation of $\mathrm{GL}_n \times \mathrm{GL}_m$. In a series of paper of Jacquet,Piatetski-Shapiro and Shalika, $L(s,\pi) := L(s, \pi_1 \times \pi_2)$ is defined as product of local Rankin-Selberg L-functions.

We will mainly focus on the case where $m = n + 1$ and $m = n$.

1.1 $m = n + 1$

For a reductive group G over F, let $[G] := G(F)\backslash G(\mathbb{A})$ be the adèlic quotient.

In this subsection we set $G = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$ and H be the "diagonal" subgroup $g \mapsto \left(g, \begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$ where $g \in \mathrm{GL}_n$

Assume that $\pi \hookrightarrow \mathcal{A}_{\mathrm{cusp}}([G])$ is cuspidal, the method to establish the analytic property of $L(s,\pi)$ is through integral representation.

Let $\varphi \in \pi$, consider

$$\mathcal{P}_H(\varphi, s) = \int_{[H]} \varphi(h) |\det h|^{s-\frac{1}{2}} dh$$

Called the *period integral* along H. When $s = \frac{1}{2}$, we write $\mathcal{P}_H(\cdot, s) = \mathcal{P}_H$

Proposition 1.1. One has the following result

(1) The integral defining $\mathcal{P}_H(\varphi, s)$ is integrable for all $s \in \mathbb{C}$, and defines an entire function of s.

(2) When $\varphi = \otimes_v \varphi_v$ is factorizable, then $\mathcal{P}_H(\varphi, s) = \prod_v I_v(\varphi_v, s)$. Let S be the union of archimedean places and the set of finite places where φ_v is not spherical, then for $v \notin S$, $I_v(\varphi_v, s) = L(\pi_v, s)$

Remark. I_v is defined by local Rankin-Selberg integral

Thus, one has

$$\mathcal{P}_H(\varphi, s) = L(s, \pi) \times \left(\prod_{v \in S} e_v(s, \varphi_v) \right)$$

where

$$e_v(s, \varphi_v) = \frac{I_v(\varphi_v, s)}{L(s, \pi_v)}$$

It is known that $e_v(s, \varphi_v)$ is holomorphic for any $\varphi_v \in \pi_v$ and there exists $\varphi_v \in \pi_v$ such that $e_v(s, \varphi_v)$ is nowhere vanishing. As a consequence

Proposition 1.2.

$$L(\frac{1}{2}, \pi) \neq 0 \iff \mathcal{P}_H|_\pi \neq 0$$

1.2 $m = n$

We introduce more notations: for reductive group G over F, let A_G^∞ be the neutral compoent of real points of split center of $\mathrm{Res}_{F/\mathbb{Q}} G$, which is isomorphic to product of positive real line(as Lie groups). Let $G(\mathbb{A})^1$ be the subgroup of $G(\mathbb{A})$ consists of g such that $|\chi(g)| = 1$ for any $\chi \in X^*(G)$. It descends to subset $[G]^1$ of $[G]$. One then has decomposition

$$G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G^\infty$$

Also denote the quotient $[G]_0 = [G]/A_G^\infty$ one has canonical isomorphism $[G]^1 \cong [G]_0$.

Let $\mathcal{S}([G])$ be the space of Schwartz function on $[G]$ and $\mathcal{T}([G])$ be the space of smooth function on $[G]$ of uniform moderate growth.

We denote $\mathbb{A}_n$ to be the rank n free module over $\mathbb{A}$ realized as row vectors, and $\mathbb{A}^n$ be the same but realized as column vectors.

Now, for this subsection, denote $G = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ and H be the diagonal subgroup $\mathrm{GL}_{n,F}$. Let $\pi = \pi_1 \times \pi_2$ be a cuspidal automorphic representation of $G(\mathbb{A})$, for simplicity we assume the central character of π is trivial on A_G^∞ (to avoid twist).

Let $\Phi \in \mathcal{S}(\mathbb{A}_n)$ be a Schwartz function on $\mathbb{A}_n$. One can then defines a Θ-seris on $[H]$:

$$\Theta(h, \Phi) = \sum_{v \in F_n} \Phi(vh)$$

Denote $\Theta'(h,\Phi) = \Theta(h,\Phi) - \Phi(0)$, then define the mirabolic Eisenstein series as

$$E(h,\Phi,s) = \int_{A_G^\infty} \Theta'(ah,\Phi)|\det(ah)|^s da$$

The expression is absolutely convergent for $\mathrm{Re}(s) > 1$, and defines a holomorphic map from $\{s : \mathrm{Re}(s) > 1\}$ to $\mathcal{T}([H])$, which has a meromorphic continuation with simple poles at $s = 0$ and $s = 1$.

For $\varphi \in \pi$, define the period

$$\mathcal{P}_{H,\Phi}(\varphi,s) = \int_{[H]_0} \varphi(h)E(h,\Phi,s)dh = \int_{[H]} \varphi(h)\Theta'(h,\Phi)|\det h|^s dh$$

When $s = \frac{1}{2}$, denote $\mathcal{P}_{H,\Phi}(\cdot,s) =: \mathcal{P}_{H,\Phi}(\cdot)$. The following proposition is a consequence of meromorphic continuation of Eisenstein series.

Proposition 1.3. We have the following assertions:

(1) $\mathcal{P}_{H,\Phi}(\varphi)$ is meromophic function of s, and it has simple poles 0 and 1 if $\pi_2^\vee \cong \pi_1$, otherwise it is entire.

(2) When $\varphi = \otimes\varphi_v$ factorizable and $\Phi = \prod \Phi_v$ factorizable. Then $\mathcal{P}_{H,\Phi}(\varphi,s) = \prod_v I_v(\varphi_v,\Phi_v,s)$. Let S be the union of archimedean places and finite places where Φ and φ is not spherical, then $I_v(\varphi_v,\Phi_v,s) = L(s,\pi)$.

Similar to the situation when $m = n+1$, I_v is local Rankin-Selberg integral and we have

Proposition 1.4.

$$L(\frac{1}{2},\pi) \neq 0 \iff \mathcal{P}_{H,\Phi}|_\pi \neq 0$$

2 GGP conjectures for unitary groups

2.1 Base change for unitary groups

Let E/F be a quadratic extension of number field. Let $\mathcal{H} = \mathcal{H}_n$ be the isomorphism classes of E/F Hermitian spaces of dimension n, and $\mathcal{H}^-$ be the isomorphism classes of E/F skew-Hermitian spaces of dimension n.

For $h \in \mathcal{H}$ or $\mathcal{H}^-$, let $\mathrm{U}(h)$ be the unitary group of h. Denote $\mathrm{GL}_{n,E} := \mathrm{Res}_{E/F}\mathrm{GL}_n$, let π be an automorphic cuspidal representation of $\mathrm{U}(h)(\mathbb{A})$. There is a notion of base change of π to $\mathrm{GL}_n(\mathbb{A}_E)$ which we now explain.

Let π be a cuspidal automorphic representation of $\mathrm{U}(h)(\mathbb{A})$, we say that an automorphic representation Π of $\mathrm{GL}_{n,E}$ is a **(weak) base change** of π if for almost all places v, the Langlands parameter of Π_v is given by $\mathrm{BC} \circ \phi_\pi$ where $\phi_\pi : \mathrm{WD}_{F_v} \to {}^L\mathrm{U}(h)_{F_v}$ is the Langlands parameter for π and $\mathrm{BC} : {}^L\mathrm{U}(h)_{F_v} \to {}^L\mathrm{GL}_{n,E_v}$ is the base change morphism.

Theorem 2.1 (Mok,Kaletha-Minguez-Shin-White)**.** For all cuspidal automorphic representation π of $\mathrm{U}(h)$, the base change (weak or strong) exists, denote by $\mathrm{BC}(\pi)$.

One can say more if we require the base change is generic.

Theorem 2.2 (Mok,Kaletha-Minguez-Shin-White)**.** For cuspidal automorphic representation π of $\mathrm{U}(h)$ whose base change is generic. Then $\Pi = \mathrm{BC}(\pi)$ is of the form $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_k$ (isobaric sum), where Π_i are distinct unitary cuspidal automorphic representation of $\mathrm{GL}_{n,E}$ with $L(s, \Pi_i, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$.

We call Π of such form a **Hermitian Arthur parameter**.

Remark. By a result of Ramakrishnan, any isobaric representation of $\mathrm{GL}_{n,E}$ is determined at components of finite split places, hence we can determine base change without invoking local Langlands correspondence for unitary group.

2.2 Bessel case

For $h \in \mathcal{H}$, let $\mathrm{U}(h)$ be the unitary group of h, we use E to denote the 1-dimension E/F hermitian spaces with Hermitian metric given by $\mathrm{Nm}_{E/F}$ and the sum is orthogonal direct sum. Finally, let $\mathrm{U}_h = \mathrm{U}(h) \times \mathrm{U}(h \oplus E)$.

For a Hermitian Arthur parameter for $\mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$ we mean the representation of the form $\Pi_n \times \Pi_{n+1}$, where each Π_k is a Hermitian Arthur parameter of $\mathrm{GL}_{k,E}$ for $k = n, n+1$.

Theorem 2.3 (Beuzart-Plessis,Chaudouard,Zydor,based on their earlier works and works of Jacquet-Rallis,Zhang,Xue,Liu,Zhu,Yun,Gordan,...)**.** Let Π be an Arthur Hermitian parameter of $\mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$. Then the following are equivalent:

(1) $L(\frac{1}{2}, \Pi) \neq 0$.

(2) There exists $h \in \mathcal{H}$ and cuspidal automorphic representation σ of U_h such that Π is a weak base change of σ and
$$\mathcal{P}_{W_h}(\varphi) = \int_{[\mathrm{U}'_h]} \varphi(h) dh$$
defines a non-zero linear form on σ.

The period above is called **Bessel period**

Remark. Low rank case reduced to the famous Waldspurger formula.

Remark. It is proved previously by Wei Zhang under some local conditions and some local condition are removed by work of Beuzart-Plessis and Hang Xue. Beuzart-Plessis-Liu-Zhang-Zhu then prove the stable case, that is under the assumption that Π is cuspidal.

2.3 Fourier-Jacobi case

For $(h, V) \in \mathcal{H}^-$, let (V_0, h_0) denote V regarded as a vector space over F together with symplectic form $\mathrm{Tr}_{E/F} h$.

Let ψ be a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$, this determines a Weil representation $\omega = \omega_\psi$ of $\mathrm{Mp}(V_0)(\mathbb{A})$. Fix a polarization $V_0 = X + Y$ where X and Y are Lagrangian, then ω is realized on $\mathcal{S}(X(\mathbb{A}))$ (Schrödinger model).

Let $\eta : \mathbb{A}^\times \to \{\pm 1\}$ be the quadratic character associated to E/F and $\mu : \mathbb{A}_E^\times \to \mathbb{C}^\times$ be an extension of η to $\mathbb{A}_E^\times$. Such μ determines a section $\mathrm{U}(h)(\mathbb{A}) \to \mathrm{Mp}(h_0)(\mathbb{A})$. Hence (ψ, μ) determines a representation of $\mathrm{U}(h)(\mathbb{A})$ on $\mathcal{S}(X(\mathbb{A}))$. For $\phi \in \mathcal{S}(X(\mathbb{A}))$ we can form the theta series as a function on $[\mathrm{U}(h)]$:

$$\theta(g, \phi) = \sum_{x \in X(F)} \omega(g)\phi(x)$$

Let $\mathrm{U}_h = \mathrm{U}(h) \times \mathrm{U}(h)$ and U'_h denote the diagonal subgroup of U_h. For a cuspidal automorphic representation σ of U_h and $\varphi \in \pi$. The **Fourier-Jacobi period** is defined as

$$\mathcal{P}_{\mathrm{U}'_h, \phi}(\varphi) = \int_{[\mathrm{U}'_h]} \varphi(h)\overline{\theta(h, \phi)} dh$$

Theorem 2.4. Let Π be an Arthur Hermitian parameter of $\mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$. Then the following are equivalent:

(1) $L(\frac{1}{2}, \Pi \otimes \mu^{-1}) \neq 0$.

(2) There exists $h \in \mathcal{H}^-$ and a cuspidal automorphic representation σ of U_h such that Π is a weak base change of σ and $\phi \in \mathcal{S}(X(\mathbb{A}))$ such that $\mathcal{P}_{\mathrm{U}'_h, \phi}$ is non-zero on σ.

This is an ongoing joint work with Paul Boiseau and Hang Xue, it has been proved under some local assumption by the work of Hang Xue.

3 Relative trace formulas and proofs of global GGP

GGP conjectures look very similar to Rankin-Selberg theory, indeed it is not only the motivation for proposing GGP but also the proof: compare unitary group to general linear group via relative trace formula (RTF).

3.1 Quick review of trace formula

Some notation: for reductive group G over F, $\mathcal{S}(G(\mathbb{A}))$ denote the space of Schwartz function on $G(\mathbb{A})$ which is compact supported in non-archimedean place and rapid decreasing in archimedean place.

For simplicity, let's assume G is anisotropic, so that $[G]$ is compact, as a consequence $L^2([G])$ decomposes into Hilberg direct sum of (cuspidal) automorphic representations.

$G(\mathbb{A})$ acts on $L^2([G])$ via right translation, for For $f \in \mathcal{S}(\mathbb{A})$ this induces a right translation $R(f)$ of $\mathcal{S}(G(\mathbb{A}))$ on $L^2([G])$. The operator $R(f)$ is of trace class. The trace formula in this situation is just compute $\mathrm{tr}\, R(f)$ in two different ways.

- Spectrally: for each irreducible component π of $L^2([G])$, the trace of $R(f)$ is given by

$$\mathrm{tr}\, R(f)|_\pi = \sum_{\varphi \in \mathcal{B}_\pi} \langle R(f)\varphi, \varphi \rangle$$

- Geometrically: The action of $R(f)$ is given by the kernel function

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

called the **automorphic kernel function**, hence the trace of $R(f)$ is given by

$$\operatorname{tr} R(f) = \int_{[G]} K_f(x,x)dx$$

Write $G(F)$ as union of conjugacy class, each conjugacy class then has contribution to trace of $R(f)$.

In order to generalize, note that in essesce, we have a decomposition not only the *trace*, but the *kernel function*.

We hace

$$K_f(x,y) = \sum_{\pi} K_{f,\pi}(x,y) = \sum_{a} K_{f,a}(x,y) \tag{1}$$

where

- π runs through irreducible component of $L^2([G])$. and K_π is the kernel function of f acting on π. Concretely

$$K_\pi(x,y) = \sum_{\varphi \in \mathcal{B}_\pi} R(f)\varphi(x)\overline{\varphi(y)}$$

- a runs through conjugacy class of $G(F)$. And $K_{f,a}$ is the term contributed by a.

integrate the equality (1) among diagonal subgroup yields trace formula.

A key utility of trace formuula is that one can compare trace formula associated to two different group: there geometric side are "the same", hence tells us the property of spectral side.

3.2 Jacquet-Rallis RTF and proof of Bessel case

From the equation (1), instead of integrate along diagonal subgroup $G \subset G \times G$, one can integrate along subgroup of the form $H_1 \times H_2$. Note that

$$\int_{[H_1 \times H_2]} K_\pi(h_1,h_2)dh_1dh_2 = \sum_{\varphi \in \mathcal{B}_\pi} \mathcal{P}_{H_1}(R(f)\varphi)\overline{\mathcal{P}_{H_2}(\varphi)}$$

Note that periods integral appears! The method to attack period integral via kernel function is generally called **relative trace formula**. Jacquet-Rallis developed a relative trace formula to attack GGP for $\mathrm{U}_n \times \mathrm{U}_{n+1}$: compare the unitary group to general linear group, which (by very complicated procedure) deduce the $\mathrm{U}_n \times \mathrm{U}_{n+1}$ GGP to the Rankin-Selberg theory, which we have introduced in section 1.

To be more precise, on the unitary side consider for $f^h \in \mathcal{S}(\mathrm{U}_h(\mathbb{A}))$, the kernel function K_{f^h} on $[\mathrm{U}_h] \times [\mathrm{U}_h]$, and consider the subgroup $\mathrm{U}'_h \times \mathrm{U}'_h$ of $\mathrm{U}_h \times \mathrm{U}_h$. We want define a distribution

$$J^h(f^h) = \int_{[\mathrm{U}'_h] \times [\mathrm{U}'_h]} K_{f^h}(x,y)dxdy$$

and expands it spectrally and geometrically.

- Spectrally: by Langlands spectral decomposition $L^2([\mathrm{U}_h]) = \widehat{\bigoplus}_{\chi\in\mathfrak{X}(\mathrm{U}_h)} L^2_\chi([\mathrm{U}_h])$. Where $\mathfrak{X}(\mathrm{U}_h)$ stands for the cuspidal datum of U_h. Note that any cuspidal automorphic representation π is a cuspidal datum with L^2_π consists of closures of functions in π.

  Let $K_{f^h,\chi}$ be the kernel function of $R(f)$ acting on $L^2_\chi([\mathrm{U}_h])$, thus $K_{f^h} = \sum_\chi K_{f^h,\chi}$. Then consider
  $$J^h_\chi(f^h) = \int_{[\mathrm{U}'_h]\times[\mathrm{U}'_h]} K_{f^h,\chi}(x,y)dxdy$$
  igoring the issue of convergence, we have spectral expansion.
  $$J^h(f^h) = \sum_{\chi\in\mathfrak{X}([\mathrm{U}_h])} J^h_\chi(f^h)$$
  and when χ is represented by a cuspidal automorphic representation π
  $$J^h_\chi(f^h) = J^h_\pi(f^h) = \sum_{\varphi\in\mathcal{B}_\pi} \mathcal{P}_{\mathrm{U}'_h}(R(f^h)\varphi)\overline{\mathcal{P}_{\mathrm{U}'_h}(\varphi)}.$$
  In particular,
  $$\mathcal{P}_{\mathrm{U}'_h}|_\pi \neq 0 \iff J^h_\pi \neq 0. \tag{2}$$

- Geometrically, let $\mathcal{A} = \mathrm{U}'_h\backslash\mathrm{U}_h/\mathrm{U}'_h$ be the GIT quotient. For each $a\in\mathcal{A}(F)$, let $\mathrm{U}_{h,a}$ be the fiber. Then for $x,y\in\mathrm{U}'_h(\mathbb{A})$, one has decomposition
  $$K_{f^h}(x,y) = \sum_{a\in\mathcal{A}(F)} K_{f^h,a}(x,y)$$
  let
  $$J^h_a(f^h) = \int_{[\mathrm{U}'_h]\times[\mathrm{U}'_h]} K_{f,a}(x,y)dxdy$$
  then one has geometric expansion:
  $$J^h(f^h) = \sum_{a\in\mathcal{A}(F)} J^h_a(f^h)$$

On the general linear group side, consider $G = \mathrm{GL}_{n,E}\times\mathrm{GL}_{n+1,E}$ and $H = \mathrm{GL}_{n,E}$ be the "diagonal subgroup", $G' = \mathrm{GL}_{n,F}\times\mathrm{GL}_{n+1,F}$ naturally as a subgroup of G. For $f\in\mathcal{S}(G(\mathbb{A}))$, we would like to consider the distribution on $\mathcal{S}(G(\mathbb{A}))$
$$I(f) = \int_{[H]\times[G']} K_f(h,g')\eta(g')dhdg'$$
where η is the character on $G'(\mathbb{A})$ defined by $\eta(g'_n, g'_{n+1}) = \eta(\det g'_n)^{n+1}\eta(\det g'_{n+1})^n$.

- Spectrally: for $\chi\in\mathfrak{X}([G])$, define
  $$I_\chi(f) = \int_{[H]\times[G']} K_{f,\chi}(h,g')\eta(g')dhdg'$$

then, we have spectral expansion:

$$I(f) = \sum_{\chi \in \mathfrak{X}([G])} I_\chi(f) \tag{3}$$

and when χ is represented by a cuspidal automorphic representation Π of G, we have

$$I_\chi(f) = I_\Pi(f) = \sum_{\varphi \in \mathcal{B}_\Pi} \mathcal{P}_H(R(f)\varphi)\overline{\mathcal{P}_{G',\eta}(\varphi)}$$

If Π satisfies the condition in GGP conjecture (i.e. conjugate self-dual with pole of Asai L function), then the result of Flicker-Rallis says $\mathcal{P}_{G'}$ is always non-zero. And by Rankin-Selberg, $\mathcal{P}_H \neq 0 \iff L(\frac{1}{2}, \Pi) \neq 0$

$$L(\frac{1}{2}, \Pi) \neq 0 \iff I_\Pi \neq 0 \tag{4}$$

- Geometrically, let $\mathcal{A}'$ be the GIT quotient $H\backslash G/G'$, then we have geometric expansion

  $$I(f) = \sum_{a' \in \mathcal{A}'(F)} I_{a'}(f)$$

  An amazing fact is that $\mathcal{A} \cong \mathcal{A}'$, thus we can compare the geometric side of these two RTFs to get comparison of spectral side.

The above analysis leads to a proof of Bessel GGP for the stable case (that is Π is cuspidal):

By (4) and (2), we are reduced to prove $I_\Pi \neq 0 \iff$ there exists π with $\mathrm{BC}(\pi) = \Pi$ and $J_\pi \neq 0$. However, comparing geometric side by a result of Yun-Gordan,Beuzart-Plessis (fundamental lemma) and Zhang,Xue,Choudouard-Zydor (smooth transfer and singular transder), for all $f \in \mathcal{S}(G(\mathbb{A}))$ there exists a family $f^h \in \mathcal{S}(\mathrm{U}_h(\mathbb{A}))$ indexed by $\mathcal{H}$ such that

$$I(f) = \sum_{h \in \mathcal{H}} J(f^h)$$

By the result of Beuzart-Plessis-Liu-Zhang-Zhu, this implies

$$I_\Pi(f) = \sum_{h \in \mathcal{H}} \sum_{\pi : \mathrm{BC}(\pi) = \Pi} J^h_\pi(f^h)$$

this shows the result.

To show the endoscopic case: let Π be a Hermitian Arthur parameter, associated to it we have a cuspidal datum $\chi = \chi_\Pi$. A main result of Beuzart-Plessis-Chaudouard-Zydor shows $I_\chi \neq 0 \iff L(\frac{1}{2}, \Pi) \neq 0$, the remaining argument are the same.

3.3 Liu RTF and proof of Fourier-Jacobi GGP

Yifeng Liu developed a RTF to attack the Fourier-Jacobi case of GGP and it was then developed by Hang Xue to prove Fourier-Jacobi GGP under some local conditions. The method is again compares with Rankin-Selberg theory when $m = n$.

On the unitary side, denote $\mathrm{U}_h = \mathrm{U}(h) \times \mathrm{U}(h)$, and U'_h denote the diagnal subgroup. For $f^h \in \mathcal{S}(\mathrm{U}_h(\mathbb{A}))$ and $\phi_1^h, \phi_2^h \in \mathcal{S}(X(\mathbb{A}))$ and $x, y \in [\mathrm{U}'_h]$ define

$$K_{f^h,\phi_1^h,\phi_2^h}(x,y) = K_{f^h}(x,y)\overline{\theta(x,\phi_1^h)}\theta(y,\phi_2^h)$$

We have similar geometric and spectral expansion of the integral

$$\int_{[\mathrm{U}'_h]\times[\mathrm{U}'_h]} K_{f^h,\phi_1,\phi_2}(x,y)dxdy$$

On the general linear group side, denote $G = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n,E}$, let H denote the diagonal subgroup and $G' = \mathrm{GL}_n \times \mathrm{GL}_n$, for $f \in \mathcal{S}(G(\mathbb{A}))$ and $\Phi \in \mathcal{S}(\mathbb{A}_{E,n})$, consider for $h \in [H]$ and $g' \in [G']$ the kernel function

$$K_{f,\Phi}(h,g') = K_f(h,g')E(h,\Phi)$$

and consider the geometric and spectral expansion of the integral

$$\int_{[H]\times[H']} K_{f,\Phi}(h,g')\eta(g')dhdg'.$$

The remaining recipes are the same.