

# Applications of orthogonal Shimura varieties

Fall 2023 Learning Seminar

*Time:* Mondays 10 AM - 12 PM      *Location:* Room 2-361 at MIT

We discuss some applications of orthogonal Shimura varieties and their integral models, including the Tate conjecture for K3 surfaces, Picard ranks of K3 surfaces, and the averaged Colmez conjecture.

---

- (1) (*Oct. 16*) Introduction to orthogonal (SO and GSpin) Shimura varieties over  $\mathbb{C}$  and  $\mathbb{Q}$ , relation with moduli of polarized K3 surfaces
- (2) (*Oct. 23*) Tate conjecture for hyperkähler varieties (or K3 surfaces) over number fields (or  $\mathbb{Q}$ ) via Kuga–Satake construction and Faltings’s work
- (3) (*Oct. 30*) Integral models of orthogonal Shimura varieties
- (4) (*Nov. 6*) Tate conjecture for K3 surfaces in positive characteristic via Kuga–Satake construction in mixed/positive characteristic
- (5) (*Nov. 13*) Picard rank jumps for K3 surfaces in mixed/positive characteristic I
- (6) (*Nov. 20*) Picard rank jumps for K3 surfaces in mixed/positive characteristic II
- (7) (*Nov. 27*) Averaged Colmez I
- (8) (*Dec. 4*) Averaged Colmez II
- (9) (*Dec. 11*)

- 
- Possible references for moduli of polarized K3 surfaces and  $\mathrm{SO}(2, 19)$  Shimura variety: [Deb20] (complex perspective), [Riz05] (canonical models) (or [Bin21, §4] or [Tae18]). For GSpin, possible references include the surveys in [SSTT22, §2] or [AGHMP17, §2].
  - Survey on the Tate conjecture: [Tot17]. Notes on the case of K3 surfaces over  $\mathbb{Q}$ , with relation to Shimura varieties: [Hör16]. See also [And96].
  - Possible references on integral models include [Mad16; Kis09; Kis10; Kis20; Lov17]. Almost all of the other references listed below also include surveys on integral models for orthogonal Shimura varieties.

- Reference on the Tate conjecture for K3 surfaces in odd characteristic via integral models and Kuga–Satake construction in mixed/positive characteristic: [Mad15] (and the references cited there such as [Riz10] and [Mau14, §5]). For different approaches in varying generality (with less emphasis on integral models of GSpin Shimura varieties), see Totaro’s survey from above.
- References on Picard rank jumps for K3 surfaces in mixed/positive characteristic via GSpin Shimura varieties: [SSTT22; MST22; Tay22]. The complex analogue is mentioned in the introduction in [MST22].
- References on averaged Colmez conjecture via GSpin Shimura varieties: [AGHMP17; AGHMP18]. There is also a different approach: [YZ18].
- Possible other topic: Rapoport–Zink spaces and  $p$ -adic uniformization for GSpin. See [HP17].

## References

- [AGHMP17] Fabrizio Andreatta, Eyal Z. Goren, Benjamin Howard, and Keerthi Madapusi Pera. “Height pairings on orthogonal Shimura varieties”. In: *Compos. Math.* 153.3 (2017), pp. 474–534.
- [AGHMP18] Fabrizio Andreatta, Eyal Z. Goren, Benjamin Howard, and Keerthi Madapusi Pera. “Faltings heights of abelian varieties with complex multiplication”. In: *Ann. of Math. (2)* 187.2 (2018), pp. 391–531.
- [And96] Yves André. “On the Shafarevich and Tate conjectures for hyper-Kähler varieties”. In: *Math. Ann.* 305.2 (1996), pp. 205–248.
- [Bin21] Wessel Bindt. *Hyperkähler varieties and their relation to Shimura stacks*. Phd Thesis. <https://pure.uva.nl/ws/files/54829137/Thesis.pdf>. 2021.
- [Deb20] Debarre. *Hyperkähler manifolds*. Preprint, survey article. 2020. arXiv: [1810.02087v2](https://arxiv.org/abs/1810.02087v2) [math.AG].
- [Hör16] Fritz Hörmann. *The Tate conjecture for rational K3 surfaces*. Notes. <https://fhoermann.org/K3.pdf>. 2016.
- [HP17] Benjamin Howard and Georgios Pappas. “Rapoport-Zink spaces for spinor groups”. In: *Compos. Math.* 153.5 (2017), pp. 1050–1118.
- [Kis09] Mark Kisin. “Integral canonical models of Shimura varieties”. In: *J. Théor. Nombres Bordeaux* 21.2 (2009), pp. 301–312.
- [Kis10] Mark Kisin. “Integral models for Shimura varieties of abelian type”. In: *J. Amer. Math. Soc.* 23.4 (2010), pp. 967–1012.
- [Kis20] Mark Kisin. “Integral canonical models of Shimura varieties: an update”. In: *Shimura varieties*. Vol. 457. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2020, pp. 151–165.
- [Lov17] Tom Lovering. “Integral canonical models for automorphic vector bundles of abelian type”. In: *Algebra Number Theory* 11.8 (2017), pp. 1837–1890.
- [Mad15] Keerthi Madapusi Pera. “The Tate conjecture for K3 surfaces in odd characteristic”. In: *Invent. Math.* 201.2 (2015), pp. 625–668.
- [Mad16] Keerthi Madapusi Pera. “Integral canonical models for spin Shimura varieties”. In: *Compos. Math.* 152.4 (2016), pp. 769–824.
- [Mau14] Davesh Maulik. “Supersingular K3 surfaces for large primes”. In: *Duke Math. J.* 163.13 (2014). With an appendix by Andrew Snowden, pp. 2357–2425.
- [MST22] Davesh Maulik, Ananth N. Shankar, and Yunqing Tang. “Picard ranks of K3 surfaces over function fields and the Hecke orbit conjecture”. In: *Invent. Math.* 228.3 (2022), pp. 1075–1143.
- [Riz05] Jordan Rizov. *Complex multiplication for K3 surfaces*. Preprint. 2005. arXiv: [0508018](https://arxiv.org/abs/0508018) [math.AG].
- [Riz10] Jordan Rizov. “Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic”. In: *J. Reine Angew. Math.* 648 (2010), pp. 13–67.
- [SSTT22] Ananth N. Shankar, Arul Shankar, Yunqing Tang, and Salim Tayou. “Exceptional jumps of Picard ranks of reductions of K3 surfaces over number fields”. In: *Forum Math. Pi* 10 (2022), Paper No. e21, 49.

- [Tae18] Lenny Taelman. *Complex multiplication and Shimura stacks*. Preprint. 2018. arXiv: [1707.01236v4](#) [math.NT].
- [Tay22] Salim Tayou. *Picard rank jumps for K3 surfaces with bad reduction*. Preprint. 2022. arXiv: [2203.09559](#) [math.NT].
- [Tot17] Burt Totaro. “Recent progress on the Tate conjecture”. In: *Bull. Amer. Math. Soc. (N.S.)* 54.4 (2017), pp. 575–590.
- [YZ18] Xinyi Yuan and Shou-Wu Zhang. “On the averaged Colmez conjecture”. In: *Ann. of Math. (2)* 187.2 (2018), pp. 533–638.