# Diamond Assoicated to Adic Spaces

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I will talk about the Lecture X of [SW20]

## Contents



## 1 Review of Diamond

Recall Perf denotes the sites of perfectoid spaces of characteristic $p$ with pro-étale coverings.

**Definition 1.1.** A **diamond** is a sheaf on Perf, which is of the form $X/R$, where $X$ is a perfectoid space in characteristic $p$ and $R \subset X \times X$ is an equivalence relation represented by a perfectoid space, and $R \to X$ the projections are pro-étale.

Diamonds are perfectoid version of algebraic spaces.

**Example 1.2.** Any perfectoid space in characteristic $p$ is a diamond.

## 2 Diamond Associated to Adic Spaces

Now we give the key construction in this talk, $X \mapsto X^{\diamond}$, if $X$ is an analytic space over $\mathbb{Z}_p$. This is generalization of the functor $X \mapsto X^{\flat}$ when $X$ is perfectoid.

**Definition 2.1.** Let $X$ be an adic space over $\mathbb{Z}_p$, define $X^{\diamond}$ as the functor:

$$\mathrm{Perf}^{\mathrm{op}} \to \mathrm{Sets}$$

which sends a perfectoid space $T$ to the isomorphism classes of pair $((T^\sharp, \iota), f : T^\sharp \to X)$, where $(T^\sharp, \iota)$ is an untilt of $T$ and $f : T^\sharp \to X$ is a morphism of adic spaces.

If $X = \mathrm{Spa}(R, R^+)$ is an affinoid adic space over $\mathbb{Z}_p$, then $X^\diamond$ is denoted by $\mathrm{Spd}(R, R^+)$.

It should be remark that $X^\diamond$ as a functor depends on the tilting equivalenve: for maps $\phi : S \to T$ in Perf, and $(T^\sharp, \iota)$ an untilt of $T$, then by tilting equivalence, there is a unique $S^\sharp$ (up to isomorphism), such that $S^\sharp \to T^\sharp$ tilts to $S \to T$. Then define $X^\diamond(\phi)$ by sending $(T^\sharp, T^\sharp \to X)$ to $(S^\sharp, S^\sharp \to T^\sharp \to X)$.

We will denote $S^\sharp$ above as $T^\sharp|_S$.

**Remark.**
- If $X$ is a perfectoid space, then $X^\diamond$ is a representable sheaf, represented by $X^\flat$.
- One can similarly define a category fibered in groupoid(i.e. a prestack) over Perf. But by tilting equivalence again, the pair $(T^\sharp, T^\sharp \to X)$ has no non-trivial automorphism in the groupoid $X^\diamond(T)$, hence it is a presheaf.

The main theorem is following:

**Theorem 2.2.** If $X$ is an analytic adic space over $\mathbb{Z}_p$, then $X^\diamond$ is a diamond.

The proof will occupy the whole section.

**Remark.** By definition, $\mathrm{Spd}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the functor Untilt, which is a sheaf on Perf, but not a diamond. The reason is that $\mathbb{Z}_p$ is not analytic.

To proof the theorem, we first show

**Lemma 2.3.** $X$ as in the theorem, then $X^\diamond$ is a sheaf on Perf.

*Proof.* Let $\{T_i \to T\}$ be a pro-étale cover in Perf, we need to show

$$X^\diamond(T) = \mathrm{eq}\left(\prod X^\diamond(T_i) \rightrightarrows \prod X^\diamond(T_i \times_T T_j)\right).$$

Suppose we have given $(T_i^\sharp, f_i : T_i^\sharp \to X)$ for each $i$, such that $T_i^\sharp|_{T_i \times_T T_j} = T_j^\sharp|_{T_i \times_T T_j}$, and $f_i|_{T_i \times_T T_j} = f_j|_{T_i \times_T T_j}$. Then, since Untilt is a sheaf (mentioned in Hao's talk, [SW20] Lemma 9.4.5,[Sch17] Lemma 15.1(i)). Thus, there exists a unique $T^\sharp \in$ Perfd such that $T^\sharp|_{T_i} = T_i^\sharp$. Thus we only need to show that morphisms glue: there exists a unique $f : T^\sharp \to X$ such that $T_i^\sharp \to T^\sharp \to X$ is $f_i$. This reduces to the case that $X$ is affinoid, write $X = \mathrm{Spa}(R, R^+)$ where $R$ is analytic Huber ring.

The morphisms $T_i^\sharp \to X$ gives maps $R \to \Gamma(T_i^\sharp, \mathcal{O}) = \Gamma(T_i^\sharp, \mathcal{O}_{T^\sharp_{\text{proét}}})$, since $\mathcal{O}$ is a sheaf on $T^\sharp_{\text{proétale}}$, they induceds a unique map $R \to \mathcal{O}(T^\sharp)$, thus a map $T^\sharp \to X$. $\square$

Thus we have verified $X^\diamond$ is a sheaf, to proceed proving $X^\diamond$ is a diamond, we then need to show $X^\diamond$ is covered by perfectoid.

Firstly, we do the affinoid case, we then need the following lemma.

**Lemma 2.4.** (Colmez-Faltings) Let $R$ be a Tate ring, with $p$ topolocially nilpotent, consider $R' = \varinjlim_{R_i \text{ étale over } R} R_i$, then

(1) $R'$ has a natural structure of Tate ring, and $R'$ has no finite étale cover.

(2) $\widetilde{R} := \widehat{R'}$ perfectoid, and $R \to \widetilde{R}$ is pro-étale.

*Proof.* Endow $R'$ inductive topology, then $(R')^\circ = \varinjlim R_i^\circ$. then standard argument shows it is a Tate ring and has no finite étale cover. We now check $\widetilde{R} = \widehat{R'}$ is perfectoid ring.

Firstly, we need to find $\varpi \in \widetilde{R}$ such that $\varpi^p | p$ in $\widetilde{R}^\circ$. Indeed, let $\varpi_0$ be any pseudo-uniformizer of $R^\circ$. Assume $\varpi_0 | p^N$ for large $n$, the equation $x^N - \varpi_0 x = \varpi_0$ defines a finite étale algebra over $R$, since $\gcd(x^N - \varpi_0 x - \varpi_0, Nx^{N-1} - \varpi_0) = 1$ in any residue field.(Note that $\varpi_0$ is a unit!) Let $\varpi$ be the image of $x \in R[x]/(x^N - \varpi_0 x - \varpi_0)$ in $\widetilde{R}$. Note implies $\varpi \in \widetilde{R}^\circ$,i.e. it is power bounded. (Since $\varpi$ is power bounded in $R[x]/(x^N - \varpi_0 x - \varpi_0)$ with canonical topology: its power contained in $R^\circ + R^\circ x + \cdots + R^\circ x^{N-1}$), $\varpi \in \widetilde{R}$ and $\varpi^N = \varpi_0(\varpi + 1)$ implies $\varpi$ is topologically nilpotent.

Also note that $\varpi^N + \varpi_0 \varpi = \varpi_0$ implies $\varpi$ is a unit in $\widetilde{R}$. And since $\widetilde{R}$ is complete, $1 + \varpi_0$ is unit, thus $\varpi^N | \varpi_0 | p^N$ thus $\varpi_0 | p$, since $(p/\varpi)^N$ is bounded implies $p/\varpi$ is bounded.

Now, we show that $\Phi : \widetilde{R}^\circ/\varpi \to \widetilde{R}^\circ/\varpi$ is surjective. Indeed, $R'/\varpi \cong \widetilde{R}/\varpi$. Thus suffices to verify this for $\tilde{R}'$, for each $f \in R'$, assume $f \in R_i$, then consider the equation $x^p - \varpi x = f$, which is finite étale over $R_i$, hence has a solution in $R'$. □

**Remark.** I don't know why $R'$ or $\widetilde{R}$ is uniform.

Thus $\tilde{R}$ is a perfectoid pro-étale cover of $R$. We can assume $R$ has a perfectoid Galois covering $\tilde{R}$, with Galois group $G = \varprojlim G_i$, where $\tilde{R} = \widehat{\varinjlim R_i}$, $G_i = \mathrm{Gal}(R_i/R)$. Note that the $\tilde{R}$ needs to be the Tate algebra constructed above, it could be any perfectoid pro-étale covering. For example, when $R = \mathbb{Q}_p$ one can take $\tilde{R} = \mathbb{Q}_p^{\mathrm{cycl}}$

Our final goal is to prove $\mathrm{Spd}(R, R^+) = \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)/\underline{G}$, where $\widetilde{R^+}$ is the completion of integral closure of $R^+$ in $R'$. But firstly, lets show that the right hand side is a diamond. More precisely:

**Lemma 2.5.** *$\underline{G} \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+) \to \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+) \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)$(i.e. the equivalence relation) is an injection of perfectoid spaces.*

*Proof.* We need to show that, for all algebraically closed perfectoid field $C$, $\underline{G}(C, C^+) \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)(C, C^+) \to \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)(C, C^+) \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)(C, C^+)$ is an injection of sets. That is to say $G$ acts freely on $\mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)(C, C^+) = \mathrm{Hom}((\widetilde{R}^\flat, (\widetilde{R}^\flat)^+), (C, C^+))$, we can further assume that $C$ is of characteristic $p$.

Now, assume there exists $\gamma \in G$ such that $\gamma$ fix $f \in \mathrm{Hom}((\widetilde{R}^\flat, (\widetilde{R}^\flat)^+), (C, C^+))$,hence $\gamma$ fixes $f \in \mathrm{Hom}((\widetilde{R}^\flat)^+, C^+)$ then $\gamma \in G$ fixes $W(f) : W((\widetilde{R}^\flat)^+, W(C^+))$, hence quotient $I$, for $I = \ker(W(\widetilde{R}^\flat)^+ \to R)$, we get $\gamma$ fixes $\widetilde{R}^+ \to (C^\sharp)^+$, where $(C^\sharp, (C^\sharp)^+)$ is an untilt of $(C, C^+)$ (recall the proof of tilting equivalence), invert $\varpi$, we get $\gamma$ fixes $\widetilde{R} \to C^\sharp$, hence fixes each $R_i \to C^\sharp$. Standard results of étale morphism shows $\gamma$ is id on each $R_i$, hence $\gamma = \mathrm{id}$. □

Thus we get a pro-étale equivalence relation $\underline{G} \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+) \to \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+) \times \mathrm{Spd}(\widetilde{R}, \widetilde{R}^+)$.

We then need the following lemma

**Lemma 2.6.** *$X$ is a perfectoid space and $R \subset X \times X$ is a pro-étale equivalence relation, let $Y = X/R$, then the natural map $R \to X \times_Y X$ is an isomorphism.*

*Proof.* [Sch17] Proposition 11.3(ii). □

Thus we have

**Lemma 2.7.** The map $\mathrm{Spd}(\widetilde{R},\widetilde{R}^+)\to \mathrm{Spd}(\widetilde{R},\widetilde{R}^+)/\underline{G}$ is an $\underline{G}$ torsor.

*Proof.* Base change to $\mathrm{Spd}(\widetilde{R},\widetilde{R}^+)$ and applies the previous lemma. ☐

Finally

**Proposition 2.8.** $\mathrm{Spd}(\widetilde{R},\widetilde{R}^+)/\underline{G}\cong \mathrm{Spd}(R,R^+)$. In particular, $\mathrm{Spd}(R,R^+)$ is a diamond.

*Proof.* Since we know both $\mathrm{Spd}(\widetilde{R},\widetilde{R}^+)/\underline{G}$ and $\mathrm{Spd}(R,R^+)$ are sheaves on Perf, we only need to show they concide on $\mathrm{Spa}(S,S^+)$ where $S$ is a characteristic $p$ perfectoid ring. i.e., for $X=\mathrm{Spa}(S,S^+)$ perfectoid of characteristic $p$, we need to give a bijection

$$\{\tilde{X}\to X \text{ a } \underline{G}\text{-torsor}, \tilde{X}\to \mathrm{Spd}(\widetilde{R},\widetilde{R}^+), \underline{G}\text{-equivariant}\}/\simeq \longrightarrow \{(X^\sharp, X^\sharp\to \mathrm{Spa}(R,R^+)\}/\simeq$$

One the one hand, given $X^\sharp=\mathrm{Spa}(S^\sharp,S^{\sharp+})$ $X^\sharp\to\mathrm{Spa}(R,R^+)$, consider $\tilde{X}^\sharp=\mathrm{Spa}(\widetilde{S^\sharp},\widetilde{S}^{\sharp+})$, where $\widetilde{S}^\sharp=S^\sharp\widehat{\otimes}_R\widetilde{R}$, gives an object in the left hand side. Coversely, given an object in the left hand side, it descends to an object in the right hand side. More specifically, $\tilde{X}\times_X\tilde{X}\cong X\times\underline{G}$. And a maps $\tilde{X}\to\mathrm{Spd}(\tilde{R},\tilde{R}^+)$ gives a pair $(X^\sharp, X^\sharp\to\mathrm{Spa}(\tilde{R},\tilde{R}^+)$. The action $\underline{G}\times\tilde{X}\to\tilde{X}$ gives an element in $\mathrm{Untilt}(X\times\underline{G})$ by tilting equivalence, $G$-equivariant means the the projection $\underline{G}\times X\to X$ gives the same element. Thus, by equalizer diagram

$$\mathrm{Untilt}(X)\to\mathrm{Untilt}(\tilde{X})\rightrightarrows\mathrm{Untilt}(\tilde{X}\times_X\tilde{X})$$

We get an element in $\mathrm{Untilt}(X)$. The by $G$-equivariancy, we get the desired descended morphism. ☐

Finally, we prove theorem mentioned in the beginning of this section

*Proof.* Affinoid case is proved above, general case is proved by glueing. ☐

We mention another useful property in Scholze's paper [Sch17].

**Proposition 2.9.** If $Y$ is a sheaf on Perf, suppose there is $X\in$ Perf with pro-étale morphism $X\to Y$, then $Y$ is a diamond.

*Proof.* [Sch17] Proposition 11.5. ☐

We give an example

**Example 2.10.** Let $k$ be a non-arhimedean field of charateristic $(0,p)$, denote $\mathbb{D}_k$ the open disc and $\mathbb{D}_k^*$ denotes the punctured open disc. Consider $\mathbb{D}_k\to\mathbb{D}_k : x\mapsto(1+x)^p-1$. Let $\widetilde{\mathbb{D}}_k=\varprojlim\mathbb{D}_k$.

We prove that there is an isomorphism $\mathrm{Spd}(\mathbb{Q}_p)\times\mathrm{Spd}(\mathbb{Q}_p)\cong(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p})^\diamond/\underline{\mathbb{Z}_p^\times}$

Since $\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}\to\mathrm{Spd}\,\mathbb{Q}_p$ is $\underline{\mathbb{Z}_p^\times}$ torsor, so does its base change $(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p^{\mathrm{cycl}}})^\diamond\to(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p})^\diamond$. Thus suffices to show there is a $\underline{\mathbb{Z}_p^\times}\times\underline{\mathbb{Z}_p^\times}$ equivariant isomorphism

$$(\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p^{\mathrm{cycl}}})^\diamond\cong\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}\times\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}.$$

This follow from the fact $\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}\cong\mathrm{Spa}\,\mathbb{F}_p((t^{1/p^\infty}))$.

## 3 Diamond Associated to Rigid Spaces

For a rigid space, we mean an adic space over a non-archimedean field $K$, locally of topologically finite type over $\mathrm{Spa}(K, K^\circ)$.

If $X$ is a rigid space over a non-archimedean field $K$ of characteristic $(0, p)$, we will illustrate the following:

$$\diamond : \{\text{Rigid Spaces over } K\} \to \{\text{Diamonds over } \operatorname{Spd} K\}$$

is analogous to

$$|\cdot| : \{\text{Analytic Spaces over } \mathbb{C}\} \to \{\text{Topological Spaces}\}$$

We will show

**Proposition 3.1.** Let $f : X \to X'$ be a universal homeomorphism of analytic space over $K$, then $f^\diamond : X^\diamond \to X'^\diamond$ is an isomorphism.

and

**Proposition 3.2.** The underlying topoloical space $|X|$ can be reconstructed from $X^\diamond$.

First, let's say some words on Proposition 3.1. Firstly, we have the following fact, analogues in scheme theory:

**Lemma 3.3.** Let $f : X \to X'$ be adic spaces in characteristic 0, then $f$ is a universal homeomorphism if and only if $f$ is a homeomorphism and induceds isomorphism on residue fields.

*Proof of Proposition 3.1.* It suffices to prove, by definition, for all perfectoid space $Y$ (which arises as untilts, hence not necessarily characteristic $p$, indeed it lies over $X$, hence charateristic must be 0), any morphisms $f : Y \to X'$ uniquely passes through $X$. Note that we already have a map on underlying topological space $|f| : Y \to X$, thus we only need to find map of sheaves $|f|^*\mathcal{O}_X \to \mathcal{O}_Y$. Then, we can assume $Y$ and $X$ are affinoid.

Note that residue field of $X$ and $X'$ are isomorphic implies $\mathcal{O}^+_{X'}/p^n \to \mathcal{O}^+_X/p^n$ are isomorphic as sheaves. And $Y$ is perfectoid implies $\mathcal{O}_Y = \varprojlim \mathcal{O}_Y/p^n$ as sheaves by long exact sequences. Hence we get the desired map. □

Restricted to the class of seminormal rigid spaces, the "forgetful functor" $\diamond$ does not lose information:

**Definition 3.4.** A ring is called **seminormal** if $a^2 = b^3$ implies there exists a unique $c$ such that $a = c^3, b = c^2$. A scheme(or adic space) is called **seminormal** if it is locally a seminormal ring(Huber ring).

For more on seminormal ring, see [dJ] Tag 0EUK.

**Proposition 3.5** ([SW20] Proposition 10.2.3)**.** The functor

$$\diamond : \{\text{Seminormal Rigid Spaces over } K\} \to \{\text{Diamonds over } \operatorname{Spd} K\}$$

is fully faithful.

Now, we talk about Proposition 3.2. It fits into very general procedure: we can define the underlying topological space of a diamond.

**Definition 3.6.** Let $\mathcal{D} = X/R$ be a diamond, with $X$ perfectoid and $R$ pro-étale equivalence relation. Then define $|\mathcal{D}| = |X|/|R|$, called **the underlying topological space** of $\mathcal{D}$.

**Proposition 3.7.** $|\mathcal{D}|$ is well defined, independent of choice of $(X, R)$.

We firstly need a lemma

**Lemma 3.8.** Fiber products and products exist in the category of diamond. Products of diamond coincides with products of sheaves.

*Proof.* [Sch17] Proposition 11.4, [SW20] Proposition 8.3.7. □

*Proof of Proposition 3.7.* We follow [Sch17] Proposition 11.13.

Define $|\mathcal{D}|_{\text{in}}$, the "intrinsic space", be the equivalent classes of maps $\text{Spa}(K, K^+) \to Y$, where $K$ is a perfectoid field, $K^+$ is open bounded. And two such maps are equivalent if they are dominated by a common map.

Then [Sch17] Proposition 11.13 shows that the maps $X \to \mathcal{D}$ induces a bijection $|X|/|R|$. Then, endow $|\mathcal{D}|$ with quotient topology. Then, using lemma above to show that the topology is independent of the choice (recall that pro-étale morphisms induce quotient map on underlying topological spaces). □

One has following results:

**Proposition 3.9.** If $\mathcal{D}$ is qcqs, then $|\mathcal{D}|$ is $T_0$.

Recall that for a sheaf $\mathcal{F}$ of site, $\mathcal{F}$ is called **quasi-compact** if any surjective map of sheaves $\sqcup_{i \in I} \mathcal{F}_i \to \mathcal{F}$ has a finite subcover $\sqcup_{i \in I_0} \mathcal{F}_i \to \mathcal{F}, |I_0| < \infty$. A sheaf $\mathcal{F}$ is called **quasi-separted** if for any quasi-compact $\mathcal{G}, \mathcal{H}$, with maps $\mathcal{G} \to \mathcal{F}, \mathcal{H} \to \mathcal{F}$, the fiber product $\mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ is quasi-compact.

*Proof.* [SW20] Proposition 10.3.4. □

**Proposition 3.10.** There is a bijection of open subspaces $|\mathcal{D}|$ and open subdiamond of $\mathcal{D}$.

*Proof.* [Sch17] Proposition 11.15. □

Finally, the following proposition implies Proposition 3.2.

**Proposition 3.11.** If $X$ is analytic adic space over $\mathbb{Z}_p$, then there is a natural homeomorphism $|X^\diamond| \cong |X|$

*Proof.* Reduce to affinoid case, then covering by perfectoid ones, see [Sch17] Lemma 15.6. □

# 4 Sites on a Diamond

Some relevant definitions

**Definition 4.1.** A map $f : \mathcal{F} \to \mathcal{G}$ of sheaves on Perf is called **étale**(or **finite étale**) if $f$ is locally separated (i.e. there is an open cover of $\mathcal{F}$ such that $f$ becomes separated) and any perfectoid space $X$, $\mathcal{F} \times_{\mathcal{G}} X \to X$ is étale (or finite étale).

For a diamond $Y$, define $Y_{\text{ét}}$ be the site of diamonds étale over $Y$.

**Theorem 4.2.** If $X$ is an analytic adic space over $\mathbb{Z}_p$, then $X \mapsto X^\diamond$ induces an equivalence of sites $X_{\text{ét}} \cong X^\diamond_{\text{ét}}$, which also induces $X_{\text{fét}} \cong X^\diamond_{\text{fét}}$.

*Proof.* [Sch17] Lemma 15.6. □

Thus we can translate étale cohomology of adic spaces as in Huber's work to the étale cohomology of diamond in Scholze's work [Sch17].

There is similarly a notion of quasi-pro-étale site.

# 5 Local System on Rigid Analytic Spaces

**Theorem 5.1.** Let $K$ be a complete algebraically closed extension of $\mathbb{Q}_p$, $(K, \mathcal{O}_K)$ Huber pair,$X \to Y$ is proper smooth morphism of rigid-analytic spaces over $(K, K^+)$. Let $\mathbb{L}$ be an étale $\mathbb{F}_p$-local system, then $R^i f_* \mathbb{L}$ is an étale $\mathbb{F}_p$ local system on $Y$.

We focus on the case when $Y = \text{Spa}(K, \mathcal{O}_K)$ is final. That is, we want to prove

**Theorem 5.2.** Let $X$ be a proper smooth adic space over $\text{Spa}(k, K^+)$. Then $H^i(X_{\text{ét}}, \mathbb{L})$ is finite dimensional over $\mathbb{F}_p$ for $i \geq 0$ and vanish for $i > 2 \dim X$.

Some definitions, following [Sch12].

**Definition 5.3.**

- Let $X$ be a locally Noetherian adic space, $U \in \text{Pro} - X_{\text{ét}}$ is called **pro-étale** if there exists a presentation $U = \varprojlim U_i$, with $U_i \to X$ étale and transition morphisms $U_i \to U_j$ is finite étale for $i, j$ large enough.
- $U \in X_{\text{proét}}$ is called **affinoid perfectoid** if $U$ has a pro-étale presentation $U = \varprojlim U_i$, such that $U_i = \text{Spa}(R, R^+)$ is affinoid, and $\widehat{\varinjlim (R_i, R_i^+)}$ is perfectoid ring.

**Proposition 5.4** (Colmez)**.** Let $K$ be a perfectoid field, $X$ be a locally Noetherian adic space over $\text{Spa}(K, K^+)$, then $U \in X_{\text{proét}}$ which are affinoid perfectoid form a basis for the pro-étale topology.

Affnoid perfectoid is similar to contractible open, which has cohomology vanishing property

**Lemma 5.5.** Let $X$ be a locally Noetherian adic space, $U \in X_{\text{proét}}$ is affinoid perfectoid. Then $H^i(U, \mathbb{L} \otimes \mathcal{O}_X^+/p)$ is almost zero.

As a consequence

**Lemma 5.6.** Let $K$ be a algebraically closed complete extension of $\mathbb{Q}_p$, and let $X$ be a proper smooth adic space over $(K, \mathcal{O}_K)$. Then $H^i_{\text{ét}}(X, \mathbb{L} \otimes \mathcal{O}_X^+/p)$ is almost finitely generated $\mathcal{O}_K$ module, which is almost zero for $i > 2 \dim X$.

*Proof.* Take affinoid perfectoid cover, then mimic the proof of Cartan-Serre theorem, by some technical spectral sequence and Cech cohomology argument. □

Now we can prove the Theorem 5.2.

## References

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