Algebraization of Complex Torus

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I will talk about the Section 1.3 of [1]

Contents

1 Main Theorem

The main theorem today will be about when a complex torus is algebraic (i.e. has a projective embedding into $\mathbb{CP}^n$.

Recall that let V be a complex vector space of dimension g and $U \subset V$ is a lattice (free abelian group of rank $2g$ which span V as a $\mathbb{R}$-vector space). Then $X = V/U$ is a complex torus, which is a complex compact Lie group and a complex manifold of Kähler type. V and U can be canonically read from X:$V = T_0X, U = H_1(X;\mathbb{Z})$.

Also recall that $\mathrm{NS}(X) \subset H^2(X;\mathbb{Z})$ consists of Chern classes of line bundle, which can be identified with Hermitian form on V with imaganary part integral on U.

The main theorem today is

Theorem 1.1. X has an embedding into $\mathbb{CP}^n \iff$ there exists positive definite $H \in \mathrm{NS}(X)$.

Any such H is called **polarization** for X.

Note that existence of such H is equivalent to an alternating bilinear form E on V such that $E(ix, iy) = E(x, y)$, $E(ix, y)$ is positive definite. Such E is called **Riemann form** for such (V, U)

By Chows theorem or GAGA, this holds if and only if there exists a projective variety A such that $X = A(\mathbb{C})$. By GAGA again, A has an group scheme structure, which is smooth since X is. Thus, as a corollary

Corollary 1.2. $X = A(\mathbb{C})$ for some abelian variety over $\mathbb{C}$ if and only if there exists a positive definite $H \in \mathrm{NS}(X)$.

We also have

Corollary 1.3. Any complex torus is (complex points of) an elliptic curve, i.e. an embedding into $\mathbb{CP}^n$.

Proof. Proof 1: Take any Hermitian metric h on $\mathbb{C}$, let e_1, e_2 be a basis for the lattice, then $h/h(e_1, e_2)$ is a Riemann form.

Proof 2: Let $\wp$ be the Weierstrass $\wp$-function for the torus.

$$\wp(z) = \sum_{u \in U, u \neq 0} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2} \right)$$

Proof 3: Any compact Riemann surface is algebraic. (Using Kodaira embedding below, or Riemann-Roch).

□

Outline of today's talk:

- Prove the main theorem more cleanly by Kodaira embedding theorem.
- Prove the main theorem with hands dirty by analyzing line bundles on X, which also provides more information.

2 First proof: Kodaira Embedding Theorem

We call a compact manifold **projective** if it can be embedded into $\mathbb{CP}^n$, let's recall the theorem:

Theorem 2.1 (Kodaira embedding)**.** Let X be a compact complex manifold of Kähler type, then X is projective if and only if there exists a positive holomorphic line bundle on X.

As a corollary, (together with Lefschetz 1-1 theorem),

Corollary 2.2. Let X be a compact complex manifold, then X is projective if and only if X has a Kähler metric whose fundamental form ω represents an integral cohomology class, i.e. $[\omega] \in H^2(X, \mathbb{Z})$.

Let $\mathcal{K}_X \subset H^2(X; \mathbb{R})$ be the **Kähler cone**, i.e. a cohomology class which is represented by a Kähler metrics.

Lemma 2.3. Let $X = V/U$ be a complex torus, then $E \in H^2(X; \mathbb{R}) \cong \mathrm{Alt}(V \times V, \mathbb{R})$ is a Kähler form if and only if E is imaginary part of a positive Hermitian form on V (i.e. E is a positive (1,1) form).

Proof. On the one hand, if E is an alternating forms on V which is imaginary part of a Hermitian form, let H be the Hermitian form, then H defines a Kähler metric on V which descends to a Kähler metric on X.

On the other hand, let g be a Kähler metric on X, with Kähler form ω, let $\tilde{g}$ be the average of g on X with respect to Haar measure:

$$\tilde{g}(V,W) = \int_X g((l_x)_*V,(l_x)_*W)dx$$

Then the fundamental form of $\tilde{g}$ of average of ω, hence also closed. Thus $\tilde{g}$ is also a Kähler metric, which is translation invariant. Moreover, $[\omega] = [\int_X \omega]$ by Hodge decomposition (note that Harmornic form with respect to constant metric is constant form). Thus any $[\omega]$ in the Kähler cone is a positive (1,1) form. $\square$

Now we show our main theorem 1.1.

Proof of Theorem 1.1. X is projective $\iff$ $H^2(X;\mathbb{Z}) \cap \mathcal{K}_X \neq \varnothing$. However, $H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{R})$ is exactly $\mathrm{Alt}(U \times U, \mathbb{Z})$. $\square$

We discuss a briefly about the matrix interpretation of above theorem. Assume $V = \mathbb{C}^g$, pick a $\mathbb{Z}$-basis for U, say $e_1, \cdots, e_{2g}$, the $g \times 2g$ matrix $\Pi = (e_1, \cdots, e_{2g}) \in \mathrm{M}_{g\times 2g}(\mathbb{C})$ is called the **period matrix**. The main theorem can be interpreted as (by some linear algebra), there exists a non-degererate, alternating matrix $A \in \mathrm{M}_{2g}(\mathbb{Z})$ such that

$$\Pi A^{-1} \Pi^t = 0, i\Pi A^{-1} \overline{\Pi} > 0.$$

These two are called **Riemann relations**.

Example 2.4. Let $U \subset \mathbb{C}^2$ be the lattice with a $\mathbb{Z}$-basis $(1,0)^t, (0,1)^t, (\sqrt{-2}, \sqrt{-3})^t, (\sqrt{-5}, \sqrt{-7})^t$, then it is not algebraic.

Proof. Indeed, there is no $B \in \mathrm{GL}_n(\mathbb{Q})$, such that $\Pi B \Pi^t = 0$. $\square$

3 Global Sections of Line Bundles on X

Recall in the precious talk we know the line bundles on X has a concrete description: let $\mathcal{P} = (H, \alpha)$, H is a Hermitian metric on V whose imaginary part E is integer valued on U and α is a semicharacter with respect to H:$\alpha(u_1 + u_2) = \mathrm{e}^{\mathrm{i}\pi E(u_1,u_2)} \alpha(u_1)\alpha(u_2)$. Then the short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

can be identified with

$$0 \to \mathrm{Hom}(U, S^1) \to \mathcal{P} \to \{\text{Hermitian forms with Im } \mathbb{Z}\text{-valued on } U\}$$

We denote the additive group {Hermitian forms with Im $\mathbb{Z}$-valued on U} also by $\mathrm{NS}(X)$.

Now, we give a second way of proving the theorem. Firstly, we analysis the global section of line bundles on X.

Let L be a holomorphic line bundle on X, which is given by a pair $(H, \alpha) \in \mathcal{P}$. Thus

$$H^0(X,L) = \{\theta : V \to \mathbb{C} \text{ holomorphic, } \theta(z+u) = \alpha(u) e^{\pi H(z,u) + \frac{\pi}{2} H(u,u)} \theta(z) \text{ for any } u \in U.\} \quad (1)$$

The goal is then solve this functional equation. First, we consider the case where H (or equivalently E) is degenerate.

Lemma 3.1. $N = \mathrm{rad}(H) = \mathrm{rad}(E)$ is a complex subspace of V, and $N \cap U$ is a lattice in N.

Proof. Since $H(x,y) = E(ix,y)+iE(x,y)$ thus $\mathrm{rad}(E) = \mathrm{rad}(H)$, and $\mathrm{rad}(H)$ is a complex subspace.

Clearly, $N \cap U$ is the radical of $E|_U : U \times U \to \mathbb{Z}$, and $E = (E|U)_{\mathbb{R}}$ (base extension of bilinear form), thus $N \cap U$ is a lattice of N comes from the fact the $\mathrm{rad}(B) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{rad}(B_{\mathbb{R}})$ for any alternating form B on a free abelian group. (Since for such B, $\mathrm{rad}(B)$ is a saturated subgroup, hence a direct summand, then using the usual argument of taking basis). □

By lemma above, $N/N \cap U =: Y \hookrightarrow X$ is a complex subtorus.

Lemma 3.2. Any θ in (1) is constant on cosets of N, equivalently, any section of L comes from $\bar{X} := X/Y$. (with induced α and E).

Proof. Fix $z \in V$, we show θ is constant on $z + N$. Indeed, for any $u \in N \cap U$, (1) shows $\theta(z+u) = \theta(z)$. Thus θ is holomorphic and periodic on $z + N$, thus constant by Liouville theorem (for holomorphic function of several variables). □

Thus any θ in 1 comes from V/N, and converesly any $\bar{\theta}$ on V/N gives a θ on V. Thus we reduced to the case where H is non-degenerate.

Lemma 3.3. If H is not positive definite (and non-degenerate), then $H^0(X,L) = \{0\}$

Proof. Let W be a negative subspace of H (i.e. $H|_W$ is nagetive definite), we show that θ in (1) is zero on each coset $z + W$.

Indeed, fix z, from (1), we have $|\theta(z+u)| = |\theta(z)|e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)}$, and $H(u,u)$ goes to 0 as $|u| \to \infty$ and $u \in W$, which is also the dominant term. Thus θ is bounded on $z+W$, hence constant, hence zero. □

We are then interested in the case where H is positive definite. But before proceeding, we give a lemma which is useful here and later.

Lemma 3.4 (Classification of alternating form on free abelian group)**.** Let E be an alternating form on a free abelian group A of rank $2n$ (not necessarily non-degenerate), then there exists a basis for which the matrix of E is of the form $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, where $D = \mathrm{diag}(d_1, \cdots, d_n)$ and $d_1, \cdots, d_n \geq 0$ with $d_1|d_2|\cdots|d_n, d_i \geq 0$. Moreover, such d_i are unique.

Proof. The number of 0 in D is a half of dimension of radical, thus uniquely determined, we proceed to assume E is non-degenerate.

Pick a symplectic basis of $A_{\mathbb{Q}}$, it is easily seen each $a \in A$ is integral combination of symplectic basis. Thus apply elementary factor theorem.

Uniqueness comes from computing the invariant factors. ($C = AC'B, C, C' \in \mathrm{M}_n(\mathbb{Z}), A, B \in \mathrm{GL}_n(\mathbb{Z})$, then C, C' has the same invariant factors) □

The matrix D is called **type** of E (or H or L), $\det D = \sqrt{\det E}$ is called **Pfaffian** of E, denoted by $\mathrm{Pf}(E)$, now we begin to solve the case where H is positive definite.

Theorem 3.5. When H is positive definite, $\dim H^0(X,L) = \mathrm{Pf}(E)$.

To focus on the main ideas, we first focus on a toy (but illustrating) example:

Let $V=\mathbb{C}, U=\mathbb{Z}\oplus\mathbb{Z}\mathrm{i}$, then $H(z_1,z_2)=z_1\overline{z_2}$ is polarization of V. Take $\alpha(a+b\mathrm{i})=\mathrm{e}^{\pi\mathrm{i}ab}$, it is easily check that α is a semicharacter for E. Then, the condition for θ is

$$\theta(z+u)=\alpha(u)\mathrm{e}^{\pi z\bar{u}+\frac{\pi}{2}|u|^2}\theta(z).$$

Which is equivalent to

$$\theta(z+1)=e^{\pi z+\frac{\pi}{2}}\theta(z), \theta(z+\mathrm{i})=\mathrm{e}^{-\pi\mathrm{i}z+\frac{\pi}{2}}\theta(z).$$

We want to make θ periodic, thus consider $\theta^*(z)=\theta(z)\mathrm{e}^{-\frac{1}{2}\pi z^2}$,then $\theta^*(z)$ satisfies

$$\theta^*(z+1)=\theta^*(z), \theta^*(z+\mathrm{i})=\mathrm{e}^{-\pi}\mathrm{e}^{2\pi\mathrm{i}z}\theta^*(z).$$

One can then form the Fourier expansion: write

$$\theta^*(z)=\sum_{n\in\mathbb{Z}}c_n\mathrm{e}^{2\pi\mathrm{i}nz}.$$

Then $\theta^*(z+\mathrm{i})=\mathrm{e}^{-\pi}\mathrm{e}^{2\pi\mathrm{i}z}\theta^*(z)$ shows

$$c_n/c_{n-1}=\mathrm{e}^{\pi(2n-1)}.$$

Thus $c_n=e^{-\pi n^2}$, we get that

$$\theta^*(z)=c\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\pi n^2}\mathrm{e}^{2\pi\mathrm{i}nz}.$$

for some constant c, which recovers the classical θ function

$$\theta(z,\tau)=\mathrm{e}^{\pi\mathrm{i}\tau n^2}\mathrm{e}^{2\pi\mathrm{i}nz}, (z,\tau)\in\mathbb{C}\times\mathcal{H}$$

for $\tau=\mathrm{i}$

Now we generalize this for general θ

Proof of the theorem. Choose a basis of U as in the lemma, denote U' the sublattice generated by $e_1,\cdots,e_g$. Then U' generate a maximal isotropic space V', and V' is also totally real, i.e. $V\cap iV=0$ (since H is positive definite).

Let $B=H|_{V'}$ which is a $\mathbb{R}$-valued bilinear form, extends it to a $\mathbb{C}$-bilinear form on V (since V' is totally real).

Note that $(H-B)|_{V\times V'}=0$ and $(H-B)|_{V'\times V}$ satisfies $(H-B)(z',z)=2\mathrm{i}E(z',z)$

Consider $\theta^*(z)=\mathrm{e}^{-\frac{\pi}{2}B(z,z)}\theta(z)$, writing $\alpha(u)=\mathrm{e}^{2\pi\mathrm{i}\lambda(u)}$ for some linear function $U\to\mathbb{R}$, then θ^* satisfying the funtional equation

$$\theta^*(z+u)=\mathrm{e}^{2\pi\mathrm{i}\lambda(u)}\mathrm{e}^{\pi(H-B)(z,u)+\frac{1}{2}\pi(H-B)(u,u)}\theta^*(z) \tag{2}$$

Thus $\mathrm{e}^{-2\pi\mathrm{i}\lambda(z)}\theta^*(z)$ is U'-periodic, we can form then Fourier expansion: let $(U')^\vee=\mathrm{Hom}_{\mathbb{Z}}(U',\mathbb{Z})$ be the Pontryagin dual of V'/U', then

$$\theta^*(z)=\sum_{\chi\in(U')^\vee}c_\chi\mathrm{e}^{2\pi\mathrm{i}(\chi(z)+\lambda(z))} \tag{3}$$

Note that any $u \in U$ defines an element $\hat{u} \in (U')^\vee$ by $\hat{u}(u') = E(u', u)$. Plug into the functional equation into Fourier expansion, we see

$$c_\chi = \alpha(u) e^{i\pi \hat{u}(u) - 2\pi i(\chi(u) + \lambda(u))} c_{\chi - \hat{u}}$$

And conversely, any such $\{c_\chi\}$ defines a θ^* satisfy the equation. (Coverge rapidly!) Thus it suffices to compute $\operatorname{coker}(U \to (U')^\vee)$, which, by the basis above, is easily seen to be $\det(D)$. □

4 Second Proof of the Main Theorem

A lemma:

Lemma 4.1 (Theorem of the Square)**.** For $a \in X, t_a$ be the translation. Then for any $L \in \operatorname{Pic}(X), t_a^* L \otimes t_b^* L \cong L \otimes t_{a+b}^* L$

Proof. Compare the associated pair (H, α). □

Corollary 4.2. $t_a^* L \otimes t_{-a}^* L = 1$,the trivial bundle.

Now, we give another proof of the main theorem 1.1.

Proposition 4.3. Let X be a omplex torus, L be a holomorphic line bundle on X, $E = c_1(X), H$ be the associated Hermitian form, if H is not positive definite, then L is not very ample, so does powers of L. (i.e. L does not define an embedding into projective space).

Proof. If H is degenerate, then any section of s is constant on a subtorus, if H has negative subspace, then L has no global section. So does positive power of L. (Since assocaited H is a positive multiple). □

The next theorem, the theorem of Lefschetz, together with the proposition above, gives another proof to the main theorem.

Theorem 4.4 (Lefschetz)**.** If $L \in \operatorname{Pic}(X)$ with H positive definite, then $L^{\otimes 3}$ is very ample.

Proof Sketch. **Step 1**: $L^{\otimes 3}$ is globally generated generated. Indeed, by theorem of the sqaure, if θ is a section of L, then for any $a, b, z \mapsto \theta(z-a)\theta(z-b)\theta(z+a+b)$ is a section of $t_a^* L \otimes t_b^* L \otimes t_{-(a+b)}^* L \cong L^{\otimes 3}$. Then, for any z, take a, b such that $\theta(z-a), \theta(z-b), \theta(z+a+b)$ all are non-zero suffices.

Step 2:$L^{\otimes 3}$ separate points. That is, for all $z_1, z_2 \in V, z_1 - z_2 \notin U$, we want a section ϕ of $L^{\otimes 3}$ such that $\phi(z_1) \neq \phi(z_2)$. We consider the ϕ of the form $\theta(z-a)\theta(z-b)\theta(z+a+b)$ for any $\theta \in$, if $L^{\otimes 3}$ does not separate points, then there exists $\gamma \in \mathbb{C}$ such that for all a, b, $\theta(z_1+a)\theta(z_1+b)\theta(z_1-a-b) = \gamma\theta(z_2+a)\theta(z_2+b)\theta(z_2-a-b)$. Then using some trick to derive a contradiction, which is explained clearly in [1].

Step 3:$L^{\otimes 3}$ separate tangent vector. Assume $z_0 \in V$ such that the tangent vector $\sum_{i=1}^g \alpha_i \frac{\partial}{\partial z_i}$ maps to 0. Then there exists α_0 such that for all ϕ, we have

$$\alpha_0 \phi(z_0) + \sum_{i=1}^g \alpha_i \frac{\partial \phi}{\partial z_i}(z_0) = 0$$

That is $D(\log(\phi)) = -\alpha_0$, where $D = \sum_{i=1}^{g} \alpha_i \frac{\partial}{\partial z_i}$. Take $\phi = \theta(z-a)\theta(z-b)\theta(z+a+b)$ as above, then ,let $f = D(\log\theta)(z)$,

$$f(z_0 - a) + f(z_0 - b) + f(z_0 + a + b) = -\alpha_0$$

This shows f is linear. Then θ satisfies there exists $\alpha \in V$ such that for all $\lambda \in \mathbb{C}$, $\theta(z + \lambda\alpha) = \mathrm{e}^{c\lambda^2 + \lambda f(z)}\theta(z)$, for some $c \in \mathbb{C}$, which contradicts with the functinoal equation. $\square$

5 Some Further Remarks

Recall that **algebraic dimension** $a(X)$ of a compact complex manifold is the transcendence dimension of the field of meromorphic function.

Proposition 5.1. A complex torus of dimension g is algebraic if and only if $a(X) = g$.

Another remarks is that

Proposition 5.2. For almost all lattice $U \subset \mathbb{C}^n$, $\mathbb{C}^n/U$ is not algebraic.

We also remark that higher cohomology groups of L can also be computed, for example, we have

Proposition 5.3. If E is nondegnerate, assume the signature of H is (r, s),then $H^q(X, L) = 0$ unless $q = s$ and

$$\dim H^s(X, L) = \mathrm{Pf}(E)$$

References

[1] Mumford. *Abelian Variety.*