Huber Rings and Adic Spaces

Weixiao Lu

May 21, 2023

I will talk about the Lecture II-III of [SW20]

Contents

1 Review of Formal Schemes and Rigid Analytic Space

1.1 Formal Schemes

The main references are Chapter 10 of [Gro], and Chapter 7 of [Bos].

Let A be an admissible topological ring, we can associate a topologically locally ringed space $(\mathrm{Spf}\, A, \mathcal{O}_{\mathrm{Spf}\, A})$ to it. Where $\mathrm{Spf}\, A$ consists of open ideals ($\mathrm{Spf}\, A = \mathrm{Spec}\, A/I$ for any ideal of definition I) and $\mathcal{O}_{\mathrm{Spf}\, A} = \lim_I \mathcal{O}_{\mathrm{Spec}\, A/I}$ where I runs through all ideal of definition.

Definition 1.1. A **formal scheme** is a topologically locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ for admissible ring A.

Example 1.2. The category of scheme fully faithfully embeds into category of formal scheme, locally given by $\operatorname{Spec} A \mapsto \operatorname{Spf} A$, with A discrete topology. In generally by glueing.

Remark. For a formal scheme X comes from a scheme, $\mathcal{O}_X(X)$ needs not to be discrete. (It is called "pseudo-discrete")

Example 1.3. Let X be a scheme, $Y \hookrightarrow X$ be a closed subscheme, one can form the **formal completion** of X along Y, denoted by $\widehat{X}_Y$

For example, if $X = \mathbb{A}^n_{\mathbb{Z}}, Y = V(p)$, then $\widehat{X}_Y = \operatorname{Spf} \mathbb{Z}_p\langle x_1, \cdots, x_n\rangle$, the Tate algebra.

Example 1.4. In deformation theory, (infinitesimal) deformation functors are often represented by a formal scheme.

We will define a fully faithful functor

$$\{\text{Adic formal schemes that locally has finitely generated ideal of definition}\} \hookrightarrow \{\text{Adic Spaces}\}$$

locally given by $\operatorname{Spf} A \mapsto \operatorname{Spa}(A, A)$.

In particular, the left hand side contains all scheme, thus category of schemes fully-faithfully embeds into category of adic spaces.

This is useful, since the right hand side contains more points. See section 1.3 for a concrete example.

1.2 Several Approaches to Non-archimidean Geometry

The main references are [Con]. For a detailed exposition of analytic manifolds and p-adic Lie theory, see Part II of [Ser], for a detailed exposition of rigid analytic space, see Part I of [Bos]. The theory of Berkovich spaces is in first 3 chapters of [Berb], a shorter exposition is in [Bera].

1.2.1 k-analytic manifold

The theory of k-analytic manifold mimic the classical differential topology, and have parallel results with one learned in a first course in differentiale manifold. This theory is good when dealing with Lie theory, but it has lots of disadvantages:

(1) There are to many analytic functions (e.g. locally constant, locally polynomial)

(2) GAGA fails wildly:

Example 1.5. For all n, m, $\dim H^0(\mathbb{P}^{n,\mathrm{an}}_k, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}_k}(m)) = \infty$, $H^i(\mathbb{P}^{n,\mathrm{an}}_k, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}_k}(m)) = 0$ for $i > 0$.

1.2.2 Tate's Approach to rigid analytic space

Throughout this subsubsection, let k be a non archimedeal field (i.e. a field with absolute value which is complete and non-trivial.)

Tate's solution to get rid of too many "analytic function" is to regard the closed unit polydisc $\{x \in k^n | |x_1| \leq 1, \cdots, |x_n| \leq 1\}$ as a global object, that is only allow the power series with "radius of convergence" at least 1 be the function on it.

Definition 1.6. The **Tate Algebra** is defined by

$$k\langle T_1, \cdots, T_n\rangle = \left\{ \sum_I a_I T^I \in k[[T_1, \cdots, T_n]] \mid |a_I| \to 0 \text{ as } |I| \to \infty \right\}$$

The Tate algebra is right candidate for functions on closed polydisc, it has following properties:

Proposition 1.7. The Tate algebra $k\langle T_1, \cdots, T_n\rangle$ is

(1) a complete Tate ring (indeed a Banach algebra, under Gauss norm $|f| = \max |a_I|$). For the notion of complete Tate ring, see Section 2.4.

(2) a Noetherian, regular, factorial ring, with Krull dimension n. And for all maximal ideal $\mathfrak{m}$, $k\langle T_1, \cdots, T_m\rangle/\mathfrak{m}$ is finite extension of k.

(3) Every ideal is closed.

Proof. See [Bos] Chapter 2. □

Definition 1.8. A **k-affinoid algebra** is a k-algebra A such that $A \cong T_n/I$ for some n and $I \subset T_n$.

The affinoid algebra is an analogue of finite type algebra over a field ,thus it corresponds to "affine algebraic variety". For an affinoid algebra A, define $M(A) = \operatorname{MaxSpec} A$. For $x \in M(A), f \in A, |f(x)| \in \mathbb{R}$ is well defined. The following example (which easily follows from proposition above) shows that $M(A)$ (together with some structure sheaf) will be a reasonable candidate for study of non-archimedean geometry.

Example 1.9.
- $M(\mathbb{C}_p\langle T_1, \cdots, T_n\rangle) = (\mathbb{C}_p^\circ)^n$.
- $M(\mathbb{Q}_p\langle T_1, \cdots, T_n\rangle) \supsetneq \mathbb{Z}_p^n$.
- $M(\mathbb{C}_p\langle X, Y\rangle/(pY - X) = p\mathcal{O}_{\mathbb{C}_p}$.

$M(A)$ has a Hausdorff totally disconnected topology, but this is not satisfiable to set up function theory, instead, there is a weaker "topology" called **Tate topology**.

Some related definitions:

Definition 1.10.
- A **rational Domain** is the subset of $M(A)$ of the form

$$U\left(\frac{f_1, \cdots, f_n}{g}\right) = \{x \in M(A) | |f_i(x) \leq |g(x)| \neq 0\}$$

We want rational domains be our "principal open" parallel to $D(f)$ in the theory of schemes.

- $U \subset M(A)$ is called **admissible open** if U can be written as union $U = \cup U_i$ such that each U_i is a rational subdomain, and for all maps between affinoid algebra $A \to B$ such that the induced map $M(B) \to M(A)$ has image lies U, then the image lies in fnitely many U_i.
- For $V = \cup V_i$ all of which are admissible open, it is called an **admissible cover** if for all maps between affinoid algebra $A \to B$ such that the induced map $\phi : M(B) \to M(A)$ has image lies in V, then $\phi^{-1}(V_i)$ has a refinement by finitely many rational subsets.

For relevant notion of Laurant subdomain, Weierstrass subdomain and affinoid subdomain, and the Gerritzen-Grauert theorem, see [Con] or Chapter 3 of [Bos].

Example 1.11. For $A = k\langle T\rangle$, both $U = \{x||x| < 1\}$ and $V = \{x||x| = 1\}$ are admissible open, $M(A) = U \cup V$, but this is not an admissible cover. This show that $M(A)$ has a great chance to be "connected".

Proof. The proof is not hard, see Example 2.2.8 of [Con], based on Maximum Modulus Principle. □

The category of admissible open on a given $M(A)$ (with morphisms are inclusion) together with admissible cover forms a cite. It defines a Grothendieck topology on the category of admissible open, we say it is a G-topology on $M(A)$. And $M(A)$ is a G-space.

Theorem 1.12 (Tate's acyclicity theorem)**.** $\mathcal{O}\left(U\left(\frac{f_1,\cdots,f_n}{g}\right)\right) = A\langle T_1,\cdots,T_n\rangle/(gT_1 - f_1, gT_2 - f_2,\cdots,gT_n - f_n)$ defines a sheaf on this site, called the **structure sheaf**.

Proof. For proof and more precise statements, see Section 4.3 of [Bos]. □

Definition 1.13.

- An **affinoid space** is a G-topologized space which is isomorphism to $\operatorname{Sp} A = (M(A), \mathcal{O}_{\operatorname{Sp} A})$,
- A **rigid analytic space** is a locally ringed G-topologized space which is locally an affinoid space.

Another goal of us is to embed the category of rigid analytic space embeds into category of adic space, that is there is a fully faithful functor:

$$\{\text{rigid analytic space}\} \hookrightarrow \{\text{adic space}\}$$

which sends $\operatorname{Sp} A$ to $\operatorname{Spa}(A, A^\circ)$ and X to X^{ad} with the following properties:

(1) X^{ad} is a locally ringed space (rather than a G-topologized space) and the underlying topological space is spectral.

(2) $\operatorname{Sp} A$ as a set is a subset of $\operatorname{Spa}(A, A^\circ)$. And $U \mapsto U \cap A$ is an inclusion-preserving bijection between the sets of quasi-compact opens in $\operatorname{Spa}(A)$ and quasi-compact admissible opens in $\operatorname{Sp}(A)$, with finite covers correponding to finite admissible covers.

(3) There is an equivalence of category $\operatorname{Shv}(X) \cong \operatorname{Shv}(X^{\text{ad}})$ (Hence same sheaf cohomology).

(1) is standard results of adic space, and for proof of (2) and (3) above, see [Hub].

1.2.3 Berkovich Space

TBA

1.3 Relationship with Formal Scheme

The main references are Section 3 of [Con] and Chapter 8 of [Bos]. They contain definition of some terminology that does not define below.

Let k be a non-archimedean field, R be the ring of integers. There is a "generic fiber" functor:

$$\{\text{admissible formal } R \text{ schemes}\} \to \{\text{rigid analytic space}\}$$

Here admissible means topologically of finite presented and flat. This functor sends $\mathcal{X}$ to $\mathcal{X}_k$.

Theorem 1.14 (Raynaud)**.** The following holds:

(1) Every qcqs rigid analytic space over k has a formal model.

(2) Any two formal models of qcqs rigid analytic space are dominated by a common admissible blow up.

Note that the "generic fiber" is not the literally defined "generic fiber", since $\operatorname{Spf} R$ consists of one point whose residue field is $R/\mathfrak{m}$ instead of k. However, if we take adic point of view, the adic version of $\operatorname{Spf} R$ is $\operatorname{Spa}(R, R)$, which has a "generic point" corresponds to the usual valuation on R. And the corresponding generic fiber is the same as adic space attached to $\mathcal{X}_k$.

2 Huber ring

2.1 Valuation Spectra $\operatorname{Spv}(A)$

The references are [Wed], [Mor]

Definition 2.1. Let A be a commutative ring, a **(multiplicative) valuation** on A is a map $|\cdot| : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, such that

- $|0| = 0, |1| = 1,$
- $|ab| = |a||b|,$
- $|a + b| \leq \max\{|a|, |b|\}$

For ring A, define $\operatorname{Spv} A$ as a set to be equivalent class of valuation on A, which can be more canonically written as

$$\operatorname{Spv} A = \{(\mathfrak{p}, R_v) | \mathfrak{p} \in \operatorname{Spec} A, R_v \subset \kappa(\mathfrak{p}) \text{is a valuation subring with } \operatorname{Frac}(R_v) = \kappa(\mathfrak{p})\}$$

If we use the letter x to denote an element $|\cdot| \in \operatorname{Spv} A$, then we use $|f(x)|$ to denote $|f|$.

There is a topology on $\operatorname{Spv} A$ whose topological basis is defined by

$$U\left(\frac{f_1, \cdots, f_n}{g}\right) = \{|f_i| \leq |g| \neq 0\}$$

Example 2.2.
- If K is a field, $\operatorname{Spv}(K) = \operatorname{RZ}(K)$, the Riemann-Zariski space of K. For example, if $K = K(C)$ the function field of a geometrically connected, smooth projective curve over $\mathbb{F}_q$, then $\operatorname{RZ}(K) \cong C$ as a topological space.

- For general A, $\operatorname{Spv} A$ is "fibered over $\operatorname{Spec} A$", whose fiber is $\mathrm{RZ}(\kappa(\mathfrak{p}))$, but the topology is much more complicated.
- $\operatorname{Spv}\mathbb{Q} = \{|\cdot|_p, \eta\}$,$\eta$ is the trivial valuation, it is the generic point. In particular, we see $\operatorname{Spv}\mathbb{Q} \cong \operatorname{Spec}\mathbb{Z}$ as topological space, and is spectral.
- $\operatorname{Spv}\mathbb{Z} = \{|\cdot|_p, \eta, \eta_p\}$, where η is trivial valuation, and η_p is defined by $\mathbb{Z} \to \mathbb{F}_p \to \{1\}$, the trivial valuation of $\mathbb{F}_p$ composed with projection.

  The topology is a bit complicated: for example η_p is specialization of $|\cdot|_p$, called "horizonal specicalization", which is a geneneral construction. See [Wed]. And η is the generic point.
- There are some rank 2 points in $\operatorname{Spv} k[x, y]$

Theorem 2.3 (Huber)**.** For any ring A,$\operatorname{Spv} A$ is a spectral space and $U\left(\frac{f_1,\cdots,f_n}{g}\right)$ is quasi-compact.

Proof. [Hub] Prop 2.2. □

Definition 2.4. A topogical space is called **spectral** if the one (hence all) of the following holds:

- X is qcqs, sober and have a basis of qc opens.
- X is homeomorphic to $\operatorname{Spec} A$ for some ring A.
- $X = \varprojlim X_i$ where X_i is a finite T_0 space, which form an inverse system

Proof. It is Definition 2.3.4 of [**?**]. □

2.2 Huber Rings

General rings is not a good place to do non-archimedean geometry or formal geometry, since there are two many points. Thus we should focus on rings with topology and continuous valuations.

Definition 2.5.
- A topological ring is called a **Huber ring**, if there exists an open subring A_0 such that A_0 (with subspace topology) is I-adic for some finitely generated ideal I.
- Any such A_0 is called **ring of definition**.
- Any such I is called **ideal of deinition**(of A_0)

Example 2.6. Examples and nonexamples of Huber rings

(1) Any ring A, with a finitely generated ideal I, then A with I-adic topology is a Huber ring, the ring of definition is itself.

(2) As a special case of (1), any discrete ring is a Huber ring.

(3) Let k be a field with non-archimedean nontrivial absolute value, then k with its natural topology is a Huber ring, with $A_0 = k^\circ$ is the ring of integer, and $I = (\varpi)$ for any $0 < |\varphi| < 1$ is an ideal of defintion.

For instantce, $\mathbb{Q}$ with p-adic topology is a Huber ring, with a ring of definition $\mathbb{Z}_{(p)}$ and ideal of definition $p\mathbb{Z}_{(p)}$

(4) k be a non-archimedean field, A affinoid k-algebra, write A as $k\langle T_1, \cdots, T_n\rangle/I$, then A is Huber with a ring of definition image of $k^\circ\langle T_1, \cdots, T_n\rangle$ and (φ) the ideal of definition.

(5) Perfectoid ring which will be discussed in the future is a Huber ring.

(6) A counter example: $A = \mathbb{Q}_p[[T]]$, $A_0 = \mathbb{Z}_p[[T]]$ with (p, T)-adic topology seems to be an example, however, it is not: A with this topology is not a topological ring: $T^n \to 0$ but $T^n/p \not\to 0$. Intuitively, the open unit disc is not a affinoid algebra.

(7) Similarly, $\mathbb{Z}_p[[T]][1/p]$ is not a Huber ring.

Definition 2.7. Let A be Huber ring (indeed topological ring suffices), a valuation $|\cdot| : A \to \Gamma \cup \{0\}$ is called **continuous**, if for all $\gamma \in \Gamma$, $B(a, \gamma) = \{b \in A | |b - a| < \gamma\}$ is open.

The subset of continuous valuation is denoted by $\mathrm{Cont}(A)$.

Theorem 2.8 (Huber)**.** If A is a Huber ring, then $\mathrm{Cont}(A)$ is spectral.

Example 2.9. $\mathrm{Cont}(\mathbb{Q}_p)$ consists of one point: the p-adic valuation.

Indeed, the valuation subring is the open subring, hence contains $p^n\mathbb{Z}_p$ for some large n and contains 1, hence contains $\mathbb{Z}_p$. It must be $\mathbb{Z}_p$ since otherwise it is the whole $\mathbb{Q}_p$.

However $\mathrm{Spv}\, \mathbb{Q}_p$ contains lots of (non-interesting) point, since a theorem of Chevalley says for any field extension (for example $\mathbb{Q}_p/\mathbb{Q}$, $\mathrm{Spv}\, \mathbb{Q}_p \to \mathrm{Spv}_{\mathbb{Q}}$ is surjective.

Example 2.10. $\mathrm{Cont}(\mathbb{Z}_p)$ consists of two points: the p-adic valuation and its horizontal specialzation: $\mathbb{Z}_p \to \mathbb{F}_p \to \{0\}$.

Indeed, consider the support of this valuation, if it is (0), then it is the valuation of $\mathbb{Q}_p$, reasoning as above, if is (p), then it is induced from the valuation of $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, which must be trivial.

Example 2.11. Let k be an algebraically closed nonarchimedean field, then $\mathrm{Cont}(k\langle T\rangle)$ consists lots of points:

- (**Classical Points** or **Type I points**)The classical points: $x \in M(k\langle T\rangle) = k^\circ$ (notation from Section 1.2) gives a continuous valuation: $f \mapsto |f(x)|$.
- (**Gauss Points** or **Type II,III points**)For $x \in k^\circ$, $r \in (0, 1]$ the interval in $\mathbb{R}$, there is a Gauss point $x_r \in \mathrm{Cont}(k\langle T\rangle$, defined by

  $$|f(x_r)| = \sup_{y \in D(x,r)} |f(y)|$$

  .

  $r = 0$ recovers classical points and $r = 1$ is the Gauss norm.
- (**Type V points**) For each $x \in k^\circ$, $r \in (0, 1]$ and each sign $\pm$, there is a rank 2 points: define $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ with the lexicographic order, where $\gamma > 1$. Then for $f \in k\langle T\rangle$, write f as $\sum_{a_n}(T - x)^n$, define

  $$|f(x_{r^\pm})| = \max_n |a_n| r^n \gamma^{\pm n}$$

A detailed discussion on this will appear on later talk, together with a complete classification. However, one should point out that $\mathrm{Cont}(A)$ is not a right model for "adic closed unit disk". The reason will be clear in the next subsection.

2.3 Huber pair

Definition 2.12. Let A be a topological ring, $S \subset A$ be a subset is called **bounded** if for any neighbourhood U of 0, there exists a neighbourhood V of 0 such that $SV \subset U$.

An element is called **power bounded**, if $\{a, a^2, \cdots\}$ is a bounded subset. The subset of all power bounded element is denoted by A°.

Proposition 2.13. If A is non-archimedean ring, i.e. A has a basis of 0 consists of additive subgroup, then A° is a subring. It is a union of bounded subring. And A° is integrally closed in A.

Example 2.14. Let k be a normed field, then $k^\circ = \{x \in k | |x| \leq 1\}$.

Example 2.15. A° needs not te be a bounded subset, for example, take $A = \mathbb{Q}_p[\varepsilon]/(\varepsilon^2)$, $A^\circ = \mathbb{Z}_p + \mathbb{Q}_p x$.

Example 2.16. A ring A with I-adic topology is bounded, thus all subsets are bounded.

Similarly, A be a Huber ring, then A_0 is bounded.

A converse is also true:

Proposition 2.17. Let A be a Huber ring, A_0 be an open subring, then A_0 is a ring of definition if and only if A_0 is bounded.

Proof. Let A be Huber ring, A_0 be a ring of definition, I is an ideal of definition, A_0' be a bounded subring, suffices to show A_0' is a ring of definition.

For S, T subset of A, define $S \cdot T$ be the additive group generated by all s, t with $s \in S, t \in T$.

Assume I is genenerated by $a_1, \cdots, a_k$, then $I^n \subset A_0'$ for some n. Thus $S = \{a_{i_1} a_{i_2} \cdots a_{i_n} | i_1, \cdots, i_n \in \{1, \cdots, k\} \in A_0'$.

Let $I' = A_0' \cdot S$ be the ideal of A_0' generated by S. We show that I' is the ideal of definition of A_0'.

Indeed, $(I')^\ell$ is generated by $\{a_{i_1} \cdots a_{i_{\ell m}}\}$. Thus $(I')^l = A_0' \cdot \{a_{i_1} \cdots a_{i_{\ell m}}\} \supset I^n \supset \{a_{i_1} \cdots a_{i_{\ell m}}\} = I^{(n+1)\ell}$. And A_0' is bounded, thus for every s there exists m such that $I^s \supset A_0' I^{mn} \supset (I')^m$. Thus $(I')^l$ form a neibourhood basis of A_0'. $\square$

Corollary 2.18. Open subring of ring of definition is a ring of definition.

A relative definition that will be useful later:

Definition 2.19. A Huber ring is called **uniform** if A° is bounded.

Definition 2.20. (1) A is a Huber ring, a subring $A^+ \subset A^\circ$ is called a **ring of integral elements** if it is open and integrally closed in A.

(2) A **Huber pair** is a pair (A, A^+) where A is Huber and $A^+ \subset A$ is a ring of integral elements.

(3) The **adic spectrum** of a Huber pair is defined as

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) | |f(x)| \leq 1 \text{ for all } f \in A^+\}$$

Theorem 2.21 (Huber)**.** If (A, A^+) is a Huber pair, then $\mathrm{Spa}(A, A^+)$ is spectral (as a subspace of $\mathrm{Cont}(A)$ or $\mathrm{Spv}\, A$.

Example 2.22. $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, $\text{Spa}(\mathbb{Z}, \mathbb{Z})$(discrete topology), $\text{Spa}(\mathbb{Z}, \mathbb{Z})$ (p-adic topology), $\text{Spa}(\mathbb{Q}, \mathbb{Z}_{(p)})$:

Since $\text{Cont}(\mathbb{Q}_p)$ consists of one point, which on $\mathbb{Z}_p$ is ≤ 1, thus $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ consists of one point. Other examples are similiar.

Example 2.23. The type V point $0_{1^+} \in \text{Cont}(A)$ is not in $\text{Spa}(A, A^\circ)$. Since for $f = 1 \in k^\circ\langle T\rangle, |f(0_{1^+})| > 1$.

For a Huber pair, A^+ contains all topological nilpotents:

Definition 2.24. Let A be a topological ring, $x \in A$ is called **topologically nilpotent**, if $x^n \to 0$. The set of all topologically nilpotent element is denoted by $A^{\circ\circ}$

Example 2.25. A with I-adic topology, then $I \subset A^{\circ\circ}$, indeed $A^{\circ\circ} = \sqrt{I}$

Example 2.26. $A = \mathbb{Q}_p$, $A^{\circ\circ} = (p)$.

Proposition 2.27. In any non-archimedean ring, topological nilpotent elements is a radical ideal of A°.

Proposition 2.28. Let (A, A^+) be a Huber pair, then $A^{\circ\circ} \subset A^+$.

2.4 Tate rings and analytic Huber rings

Now we discuss a special family of Huber rings. Which is useful for us (contains all pertectoid ring)

Definition 2.29. A Huber ring is called **Tate ring** if there exists a topologically nilpotent unit. Any topologically nilpotent unit is called a **pseudo-uniformizer**

Example 2.30. Examples and non-examples of Tate rings:

(1) Any non-archimedean field is a Tate ring, more generally, any affinoid algebra over a non-archimedean field is a Tate ring.

(2) An adic ring is almost never a Tate ring. e.g. $\mathbb{Z}_p[[T]]$ is not a Tate ring, discrete ring is not a Tate ring.

(3) A perfectoid ring is a Tate ring

Remark. A complete Tate ring is (non-canonically) equivalent to a Banach ring which admits a unit in the open unit disk. The norm on a complete Tate ring is defined by

$$|a| = \inf_{n \in \mathbb{Z} | g^n a \in A_0} 2^n$$

A wider class of Huber ring which contains all Tate ring is called analytic Huber ring

Definition 2.31. A Huber ring is called **analytic**, if the topological nilpotent unit generated the unit ideal.

Proposition 2.32. The following are equivalent:

(1) A is an analytic Huber ring.

(2) For every ring of definition A_0, each of its ideal of definition generate the unit ideal.

(3) A has no open ideals.

(4) A nas no discrete nonzero topological modules.

Proof. Not deep, see Lemma 1.1.3 of [Ked]. □

An important feature for analytic rings is that open mapping theorem holds

Proposition 2.33 (Open Mapping Theorem)**.** Let A be a complete analytic Huber ring, M, N are two complete Hausdorff, first-countable topological A-modules, then any surjective map between M and N is open.

Proof. [Ked] Theorem 1.1.9. □

Remark. An analytic ring needs not to be Huber, for example, one can take $A = \mathbb{Z}\langle \frac{a}{\rho}, \frac{b}{\rho}, \frac{x}{\rho^{-1}}, \frac{y}{\rho^{-1}} \rangle/(ax+by-1)$, see Example 1.5.7 of [Ked].

2.5 Complete Huber pairs

Definition 2.34. A Huber ring is called **complete**, if it is complete as a topological ring, or equivalently, A_0 is I-adically complete, for some (or equivalently, any) ring of definition A_0 and its ideal of definition I.

A Huber pair (A, A^+) is called **complete Huber pair**, if A is complete Huber ring.

We discuss structure sheaf, complete Huber rings will be the most interesting.

Definition 2.35. For a Huber ring A, define **completion** of A, denoted as $\widehat{A}$ as completion of A as a topological ring.

Remark. The completion of topological ring is discussed in [Bou] Chapter 3.5.

As an abelian group, $\widehat{A}$ can be characterized by $\varprojlim A/I^n$, for any ideal of definition I, this contains A as a dense subring, hence get a ring structure.

Theorem 2.36. Let A be a Huber ring, A_0 be any ring of definition, I be any ideal of definition. $\widehat{A_0}$ denote the I-adic completion of A_0, then

(1) $\widehat{A_0}$ is isomorphic to the closure of A_0 in $\widehat{A}$, with subspace topology, as a topological ring. $\widehat{A_0}$ is an open subring of $\widehat{A}$.

(2) $\widehat{A_0}$ is complete with respect to $\widehat{I} = I\widehat{A_0}$-adic topology.

(3) $\widehat{A}$ is a complete Huber ring.

It should be mentioned that I is finitely generated is essential in above theorem.

For a complete Huber pair, kind of "Nullstellensatz" holds.

Theorem 2.37 (Huber)**.** Let (A, A^+) be a complete Huber pair, then

(1) $\mathrm{Spa}(A, A^+) \neq \varnothing$ unless $A = 0$.

(2) (Adic Nullstellensatz) $A^+ = \{f \in A || f(x)| \leq 1 \text{ for all } x\}$ (This holds for general Huber pair, see Corrollary III.4.4.4 [Mor])

(3) $f \in A^\times$ if and only if $|f(x)| \neq 0$ for any $x \in X$.

Moreover we have

Theorem 2.38 (Huber)**.** For any Huber pair (A, A^+), $\mathrm{Spa}(A, A^+) \cong \mathrm{Spa}(\widehat{A}, \widehat{A^+})$

The isomorphism is given by the functoriality of Spa, discussed in the next subsection.

Example 2.39. $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $\mathrm{Spa}(\mathbb{Q}, \mathbb{Z}_{(p)})$, $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$($p$-adic topology) and $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ are homeomorphic as discussed above.

2.6 Continuous homomorphism of Huber pair

Definition 2.40. Let $(A, A^+), (B, B^+)$ be two Huber pairs, a **continuous homomorphism** of (A, A^+) to (B, B^+) is a continuous ring homomorphism $f : A \to B$ which sends A^+ to B^+.

Huber pairs form a category, which plays the rule in theory of adic same (almost) the same (except there are non sheafy pair) as the category of commutative ring in the theory of schemes.

Adic spectrum then becomes a functor:

Proposition 2.41. A continuous homomorphism of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$ induces a continuous map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$, defined by $x \mapsto (a \mapsto |\varphi(a)|_x)$. This continuous map is denoted $\mathrm{Spa}(\varphi)$ or φ^*, and it is a spectral map.

3 Adic Space

Following [**?**] ,some details are from [Wed] and [Mor]

3.1 Polynomial ring, Tate Algebra and localization of Huber rings

In this subsection we almost follow [Wed] and [Mor]

To understand the section on principal opens, we need to construct the polynomial ring and localization of Huber ring. To make it more concise, we only consider polynomial ring (and Tate algebra) with one variable, but same similar construction holds for n-variables (or even infinitely many variables, see [Wed])

Convention: let S, T be two subsets of a ring, define ST be the additive group generated by all $\{st | s \in S, t \in T\}$, in particular one can define T^2 or T^n.

Definition 3.1. Let A be a Huber ring, given a finite subset $T = \{f_1, \cdots, f_n\}$, such that they generated an open ideal. There is a topology on $A[X]$ which makes X "a free variables such that $f_i x$ is power-bounded".

More precisely, define a topology on $A[X]$ as follows: for each open neighbourhood U which is a sub-additive group of 0, define

$$U[TX] = \left\{ \sum a_i x^i | a_i \in T^i U \right\}$$

Then let $U[TX]$ be the neighbourhood basis of 0 in $A[X]$, which makes $A[X]$ a Huber ring. Which has the folloing universal properties:

- $f_i X$ is power bounded for all i.
- Let $\varphi : A \to B$ be a continuous homomorphism, with B a Huber ring (indeed non-archimedean ring suffices) and $b \in B$ such that $\varphi(f_i)b$ is power-bounded, then there exists a unique continuous homomorphism $A[X] \to B$ of A-algebra, sends X to b, i.e.

$$\mathrm{Hom}_{\mathrm{cont},A}(A[X]_T, B) = \{b \in B | \varphi(f_i)b \text{ power bounded}\}$$

Example 3.2. When $T = \{1\}$, $\mathrm{Hom}_{cont,A}(A[X]_T, B) = B^\circ$.

Remark. Similar construction works for A non-archimedean ring, see [Wed].

This is a uncompleted version of Tate algebra. The complete version is as follows:

Definition 3.3. Given T as above (they generate an open ideal), then define

$$A\langle X\rangle_T = \left\{\sum a_i X^i | a_i \in T^i U \text{ for almost all } i \text{ for any } U\right\}.$$

It similarly carries a topology, satisfying the following properties:

- When A is complete, it is completion of $A[X]_T$.
- For $\varphi : A \to B$, both are complete Huber ring, then given $b \in B$ with $\varphi(f_i)b$ bounded, there exists a unique continuous homomorphism $A[X] \to B$ of A-algebra, sends x to b, i.e.

$$\mathrm{Hom}_{\mathrm{cont},A}(A\langle X\rangle_T, B) = \{b \in B | \varphi(f_i)b \text{ power bounded}\}$$

Example 3.4. When A is a non-archimedean field and $T = 1$, it recovers the classical definition of Tate algebra. When $T = p$, it represents a disc with larger radius (radius $|1/p|$).

Remark. Similar construction works for A non-archimedean ring, see [Wed].

Finally we discuss localization

Proposition 3.5. Let T as above (they generate an open ideal), $s \in A$, then there exists a topology on A_s with the following properties

- f_i/s is power bounded.
- a continuous homomorphism $\varphi : A \to B$ factor through $A_s \iff \varphi(s)$ is invertible and $\varphi(f_i)/\varphi(s)$ is power bounded.

Proof. Just take $A_s = A[X]_T/(sX - 1)$ as a topological ring. □

Definition 3.6. A_s with above topology is denoted $A\left(\frac{f_1,\cdots,f_n}{s}\right)$. The completion of $A\left(\frac{f_1,\cdots,f_n}{s}\right)$ is denoted $A\langle\frac{f_1,\cdots,f_n}{s}\rangle$

Remark. Thus, by definition $A\langle\frac{f_1,\cdots,f_n}{s}\rangle = A\langle X\rangle_T/\text{closure of } (1 - sX)$.

Thus we recover the function ring of rational domain of an affinoid subdomain (since every ideal is closed).

3.2 Sheafy pair

For a Huber pair, we now want to define a structure sheaf on $X = \mathrm{Spa}(A, A^+)$

Definition 3.7. $s \in A, T = \{f_1, \cdots, f_n\} \subset A$ be a finite subset such that T generate an open ideal in A. Define the subset

$$U\left(\frac{T}{s}\right) = U\left(\frac{f_1, \cdots, f_n}{s}\right) = \{x \in \mathrm{Spa}(A, A^+) = |f_i(x)| \leq |g(x)| \neq 0\}$$

Subsets of this form is called **rational subset**.

Remark. $U\left(\frac{T}{s}\right)$ form a topological basis for $\mathrm{Spa}(A, A^+)$, but this is not an easy result.

The next theorem says rational subset is the analogue of $D(f)$ in scheme theory. "$U(1/s) = D(s)$"

Theorem 3.8. Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ together with a map $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ such that the map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to \mathrm{Spa}(A, A^+)$$

is final among all the maps factors through U. Moreover, it is a homeomorphism onto U.

Proof. Indeed, just take $A = A\langle\frac{f_1, \cdots, f_n}{s}\rangle$ with $A^+\langle\frac{f_1, \cdots, f_n}{s}\rangle$ image of $A^+[TX]$. $\square$

The universal property in the theorem implies that the ring the $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ only depends on U, and for rational subsets $V \subset U$, there is a restriction map $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to (\mathcal{O}_X(V), \mathcal{O}_X(V)^+)$.

Now, since rational subset forms a basis of the topology, we can define a structure presheaf $\mathcal{O}_X$ on $\mathrm{Spa}(A, A^+)$,

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{rational}} \mathcal{O}_X(U)$$

Remark.
- From definition, we see that the structure presheaf on $\mathrm{Spa}(A, A^+)$ is the same as the structure sheaf on $\mathrm{Spa}(\widehat{A}, \widehat{A^+})$.
- The structure sheaf is a sheaf of sheaf of complete topological rings, as projective limits of complete rings are complete, see [Bou] Chapter 2.3.

Example 3.9. The global section of $\mathcal{O}_X$ is $\widehat{A}$. This is slightly different from the theory of schemes.

Definition 3.10. A complete Huber pair is called **sheafy** it $\mathcal{O}_X$ defined above is a sheaf of topological rings.

Theorem 3.11. A Huber pair is sheafy in the following situation:

(1) (Schemes)A is discrete.

(2) (Formal schemes)A is adic with a finitely generated ideal of definition

(3) (Rigid Spaces)A is Tate and **strongly Notherian**. That is $A\langle T_1, \cdots, T_n\rangle$ is Noetherian for all $n \geq 0$. (for example, A is a non-archimedean field)

Remark. There will be other sheafy pair(stably uniform), which will be discussed later.

3.3 Adic Space

Before make global definition of adic space, we mentions some properties of stalk of the structure presheaf

Proposition 3.12. The stalk of the presheaf $\mathcal{O}_X$ at $x \in X = \mathrm{Spa}(A, A^+)$, denoted as $\mathcal{O}_{X,x}$, has the following properties.

- It is a local ring, with a valuation $|\cdot|$ inherit from all $\mathcal{O}_X(U), x \in U$. The support of the valuation is the maximal ideal.
- The residue field of the local ring is the completion of the residue field of x.
- Continuous homomorphism of Huber pair induces local and continuous on stalks, compatible with valuation

Globalize these, we can define adic spaces:

Definition 3.13. The category $\mathcal{V}$ is a category whose object consists of triples $(X, \mathcal{O}_X, |\cdot|_x(x \in X))$. Where $(X, \mathcal{O}_X)$ is a locally topologically ringed space, and $|\cdot|_x$ is a valuation on $\mathcal{O}_{X,x}$, whose support is the maximal ideal.

A morphism $f : X \to Y$ in category $\mathcal{V}$ is a morphism of locally topologically ringed space compatible with valuations.

Remark. Without the sheaf properties, one can still define a category $\mathcal{V}^{\mathrm{pre}}$.

Definition 3.14.

- An **affinoid adic space** is an object in the category $\mathcal{V}$ which is isomorphic to $\mathrm{Spa}(A, A^+)$.
- An **adic space** is an object in the category $\mathcal{V}$ which is locally an affinoid adic space.

Example 3.15. There exists "adicfication" functors: from the category of schemes, adic Noetherian formal schemes, rigid analytic space to adic spaces.

References

[Bera] Berkovich. *Non-archimedean Analytic Spaces.*

[Berb] Berkovich. *Spectral Theory and Analytic Geometry over Non-archimidean Fields.*

[Bos] Bosch. *Lectures on Formal and Rigid Geometry.*

[Bou] Bourbaki. *General Topology, Chapter 1-4.*

[Con] Conrad. *Several Approaches to Non-archimidean Geometry.*

[Gro] Dieudonne Grothendieck. *EGAI.*

[Hub] Huber. *Continuous Valuations.*

[Ked] Kedlaya. *Sheavas, Stacks and Shtukas.*

[Mor] Morel. *Adic Spaces*.

[Ser] Serre. *Lie groups and Lie Algebra*.

[SW20] Scholze and Weinstein. *Berkeley notes on p-adic Geometry*. 2020.

[Wed] Wedhorn. *Adic Spaces*.